Double-diffusive processes in stellar astrophysics

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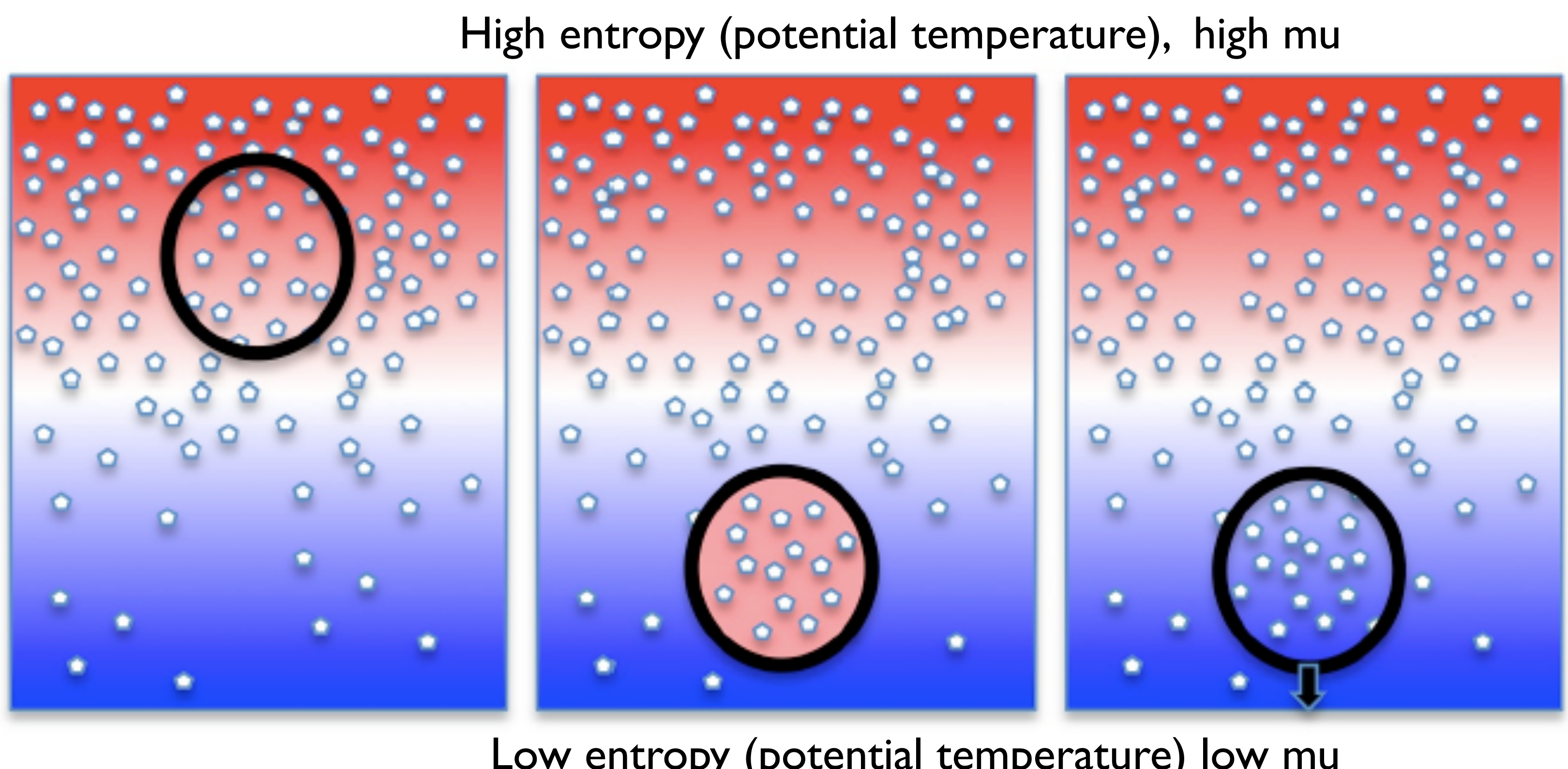
Lecture 2:
Fingering convection in stars

Recap:

Physical mechanism



Recap: Linear theory

The necessary condition for instability depends on the **density ratio**

$$R_0 = \frac{\alpha\left(T_{0z} - T_z^{ad}\right)}{\beta C_{0z}} = \frac{\delta(\nabla - \nabla_{ad})}{\phi \nabla_\mu} = \frac{\text{Stabilizing temperature stratification}}{\text{Destabilizing composition stratification}}$$

Instability to fingering occurs if

Threshold for overturning convection, Ledoux crit. →

$$1 < R_0 < \frac{\kappa_T}{\kappa_C} = \frac{1}{\tau}$$

Fastest-growing modes have

- k_z=0 (elevator modes)
- k_h ~ O(1) so wavelength O(2π)d

Stellar numbers

Typically:

- Non-degenerate regions of stars: Pr ~ 10^{-6}, $\tau \sim 10^{-7}$
- Degenerate regions of stars: Pr ~ 10^{-2}, $\tau \sim 10^{-3}$
- Finger size:

$$\sim 10d \sim 10\left(\frac{\kappa_T \nu}{N^2}\right)^{1/4} \sim 3\cdot 10^4 \left(\frac{\kappa_T}{10^7}\right)^{1/4} \left(\frac{\nu}{10}\right)^{1/4} \left(\frac{10^{-6}}{N^2}\right)^{1/4} \text{ cm}$$

- Density ratio R_0 varies substantially, and depends on mixing by fingering (e.g. Ulrich 1972; Vauclair 2004; Dennisenkov 2010)
  - In RGB stars: $R_0 \sim 10^3 - 10^6$
  - In accretion problems: $R_0 \sim 1 - 10^6$

Question: how much mixing does this instability cause?

Traditional models of fingering convection in astrophysics

Most models of mixing by fingering instabilities use a turbulent diffusivity for concentration (no mixing of temperature) of a species

$$\frac{DC}{Dt} = -\frac{1}{\rho}\nabla\cdot(\rho\mathbf{F}_C) + \frac{1}{\rho}\left(\frac{D(\rho C)}{Dt}\right)_{nucl}$$

where it is assumed that $\mathbf{F}_C = -D_C \nabla C$

where D_C is a diffusivity, and has units of cm^2/s (in cgs).

Traditional models of fingering convection in astrophysics

- Combining these equations, we get

$$\frac{DC}{Dt} = \frac{1}{\rho}\nabla\cdot(\rho D_C \nabla C) + \frac{1}{\rho}\left(\frac{D(\rho C)}{Dt}\right)_{nucl}$$

$$= \frac{1}{\rho r^2}\frac{\partial}{\partial r}\left(r^2 \rho D_C \frac{\partial C}{\partial r}\right) + \frac{1}{\rho}\left(\frac{D(\rho C)}{Dt}\right)_{nucl}$$

$$\frac{DC}{Dt} = \frac{\partial}{\partial m}\left((4\pi r^2 \rho)^2 D_C \frac{\partial C}{\partial m}\right) + \frac{1}{\rho}\left(\frac{D(\rho C)}{Dt}\right)_{nucl}$$

- (assuming that a diffusive model is appropriate….)

The only question left is:

What is D_C?

Traditional models of fingering convection in astrophysics

- The total diffusion coefficient D_C is the sum of
  - Basic atomic and collisional processes (i.e. microscopic)
  - Turbulent processes (i.e. macroscopic). For fingering only:

$$D_C = \kappa_C + D_{fing}$$

- Since D_{fing} has units of length^2/time, or length x velocity, we often (not always) estimate it from

$$D_{fing} \propto v_{fing} l_{fing}$$

Characteristic velocity of finger

Characteristic lengthscale of finger

Traditional models of fingering convection in astrophysics

- Ulrich (1972) was first to propose a mixing model for fingering convection in stars.
  - He used $l_{fing} \sim d$
  - He used $\upsilon_{fing} \sim \lambda d \sim \frac{\kappa_T}{d^2(R_0 - 1)} d$ (derive on board)
- Diffusion coefficient is therefore $D_{fing} = C_U \frac{\kappa_T}{R_0 - 1}$

with constant he argues is

$$C_U = \frac{8\pi^2\chi^2}{3}$$

where χ is the aspect ratio of finger (height / width), he argues is ~ 5 or more, so $C_U \sim 700$

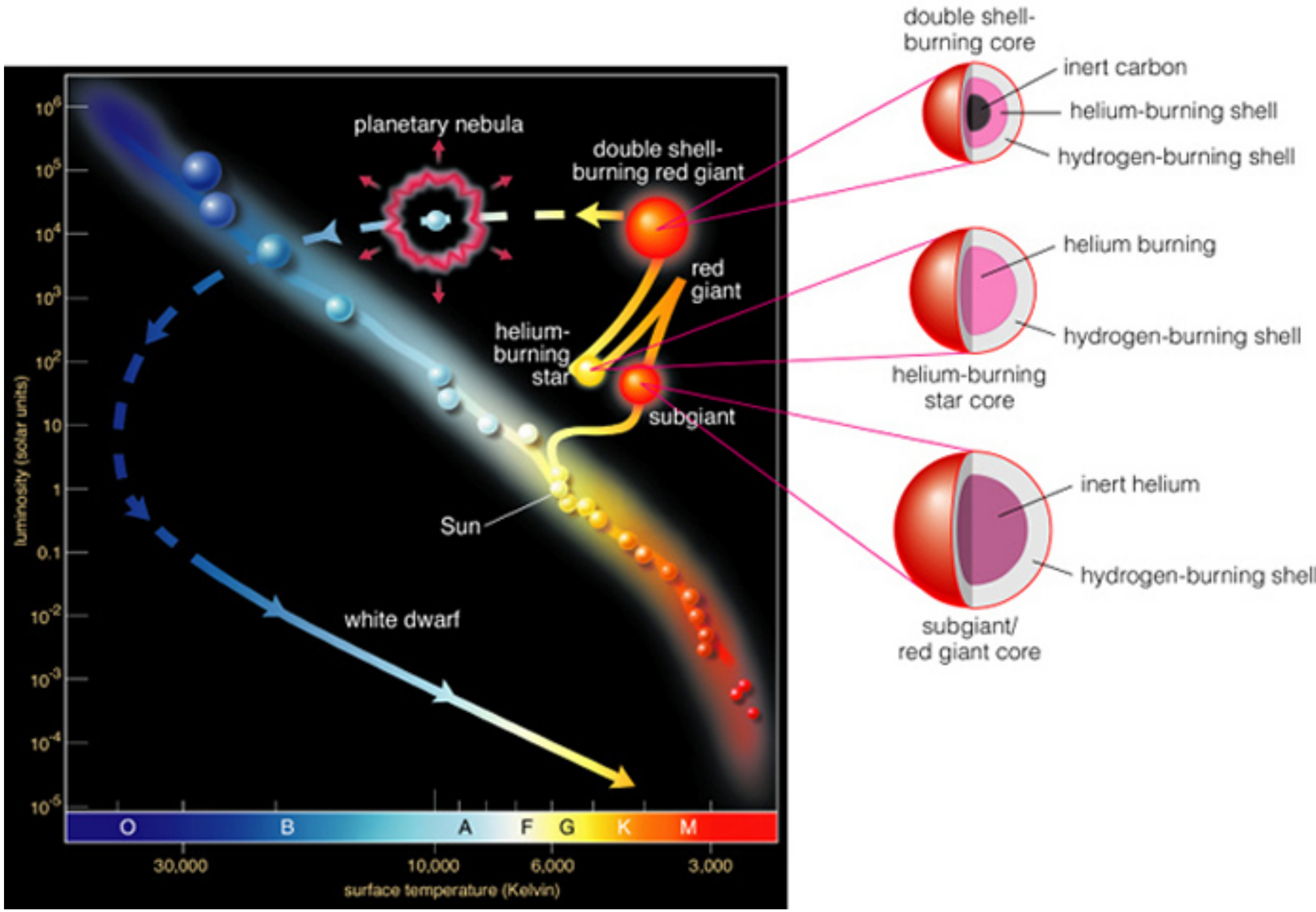
- Kippenhahn et al. (1980) arrive at similar formula, with different constant

$$D_{fing} = C_{KRT} \frac{\kappa_T}{R_0}$$

where $C_{KRT} \sim 12$

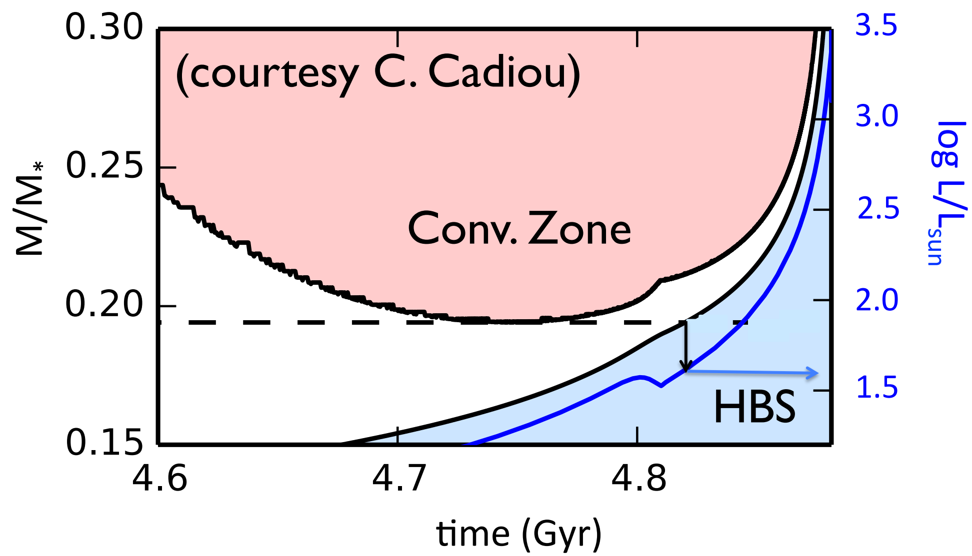
Observational constraints on the models

Fingering in RGB stars



Conventional mixing on the RGB

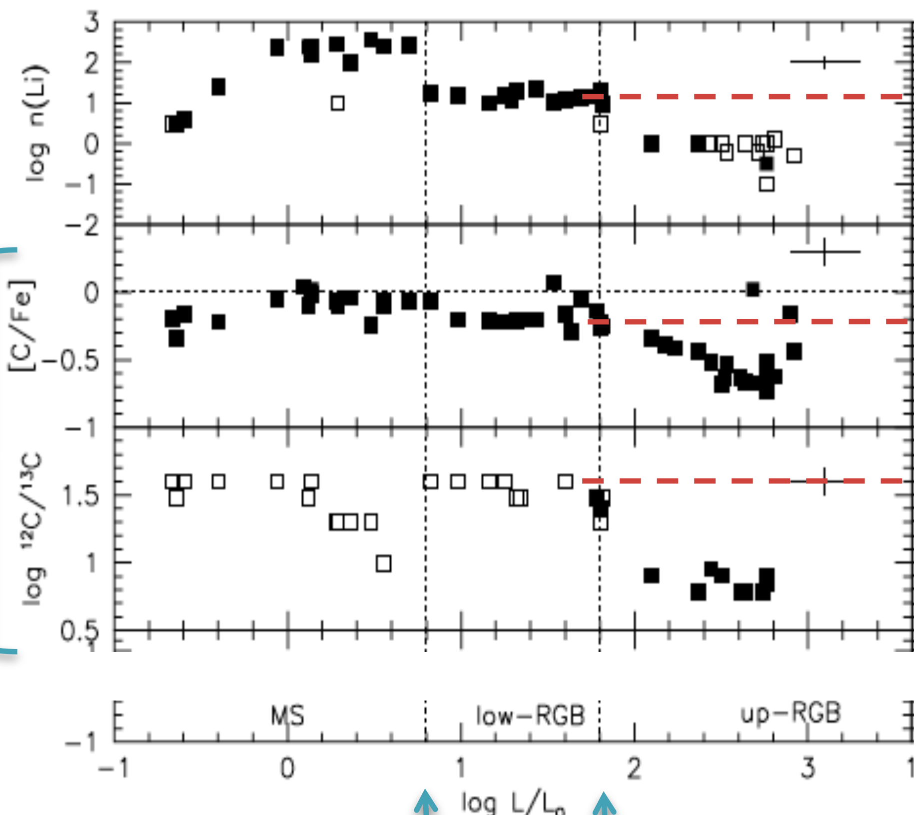
- Upon leaving the MS, the star's outer convection zone deepens and dredges up material from deep within the star: first dredge-up.



- After this event, the base of the convection zone retreats again as the Hydrogen Burning Shell moves outwards.
- The two never overlap: no more changes in surface element abundances are expected on the RGB after 1st dredge up.

Evidence for missing mixing on the RGB

Gratton et al. 2000



Burned at depth

Participate in CNO cycle at depth

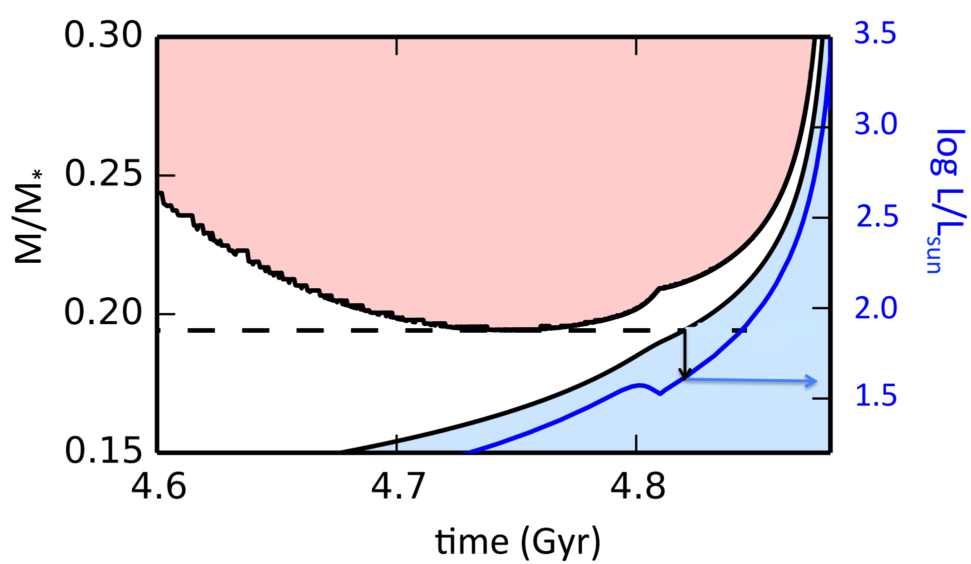
Expected levels *without* mixing post dredge-up

First dedge-up

More mixing?

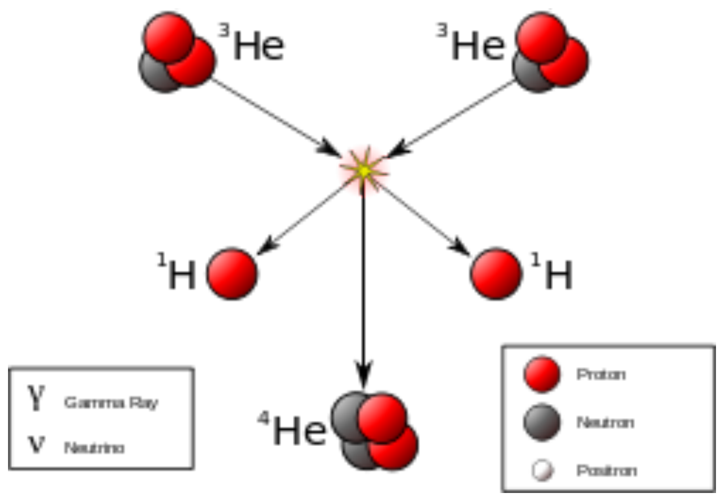
Fingering convection as the missing mixing

- The second change in surface abundances coincides with time when the hydrogen-burning shell passes through lowest-excursion point of first dredge-up = luminosity bump. Coincidence? No! (Charbonnel & Zahn 2007)



Fingering convection as the missing mixing

- Near the colder, outer edge of the hydrogen burning shell, the dominant reaction is



2 particles of total mass 6

↓

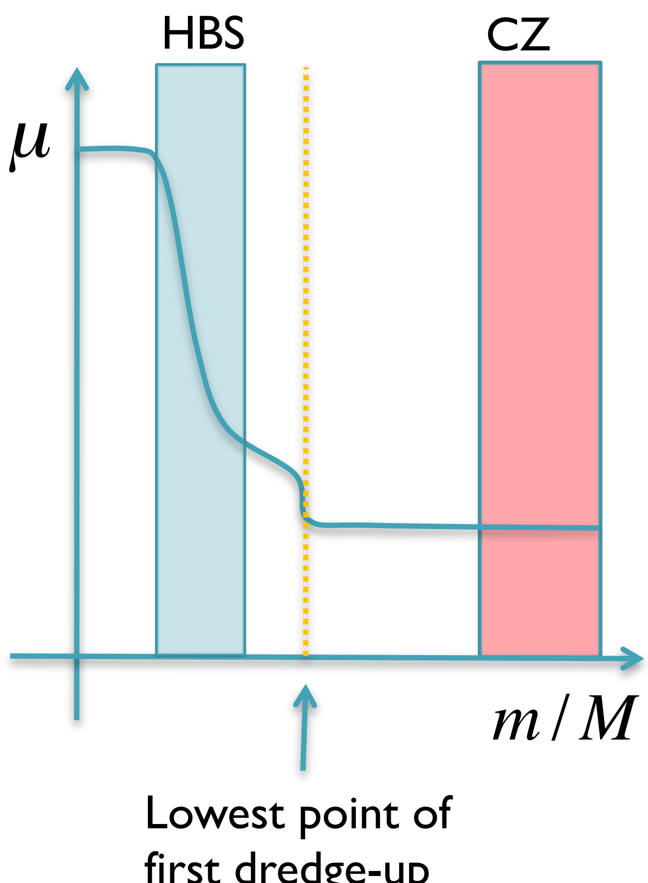
3 particles of total mass 6

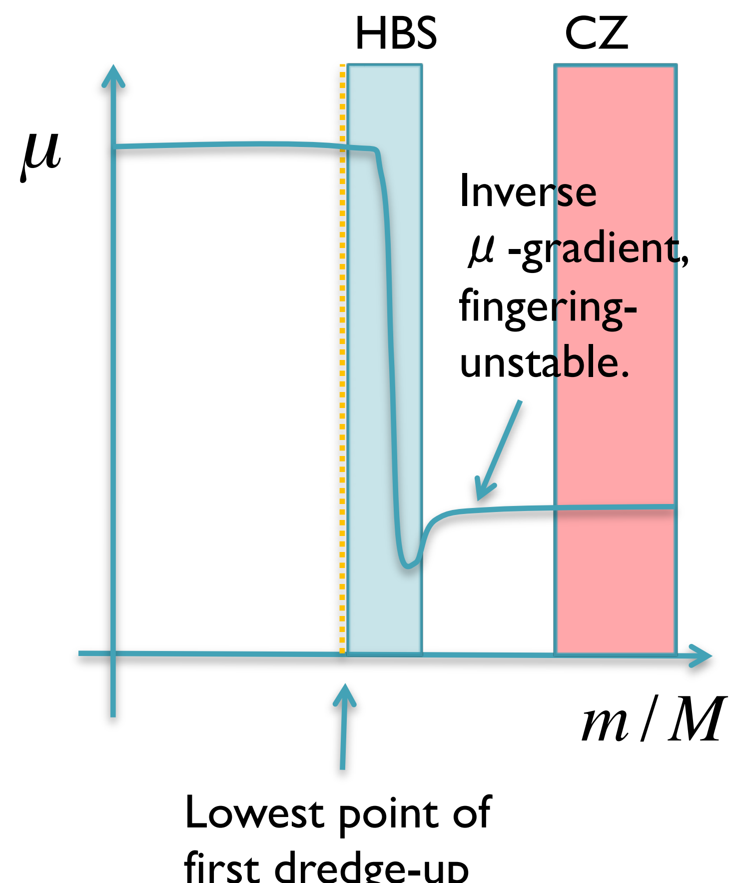
(Source: Wikipedia)

This reaction locally *decreases* the mean molecular weight (Ulrich 1972)

Fingering convection as the missing mixing

- As a result, an inverse μ–gradient forms after luminosity bump (but not before) (Charbonnel & Zahn 2007)

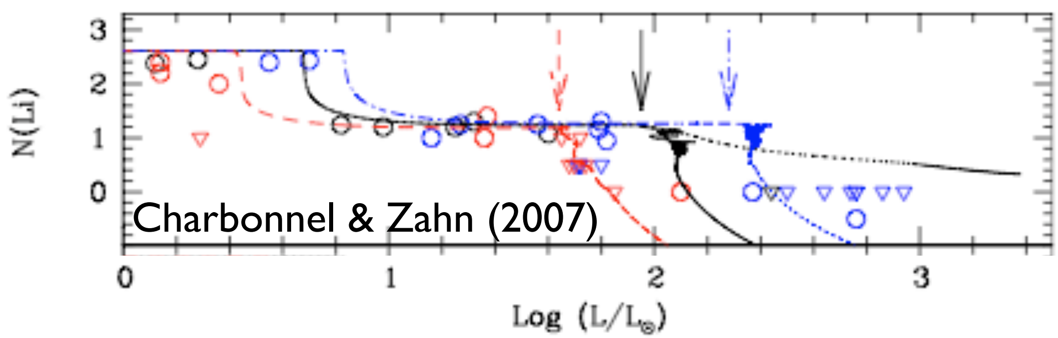




Charbonnel & Zahn (2007)

- Charbonnel & Zahn (2007) proposed that fingering convection could explain the RGB abundance observations
- They used the models of Ulrich (1972), Kippenhahn et al. (1980) for mixing coefficient:

$$D_{fing} = \frac{C}{R_0}\kappa_T \quad \text{where} \quad R_0 = \frac{\delta}{\phi}\frac{\nabla - \nabla_{ad}}{\nabla_\mu}$$



Black solid line: C = 1000

Black dotted line: C = 100

The C = 1000 value is consistent with prediction by Ulrich (1972) and explains RGB observations…

✔ All good! …. (or is it?)

Direct numerical simulations (DNSs) of fingering convection

Numerical simulations as experimental tool

- The last two decades have seen the emergence of supercomputing as an experimental tool in astrophysics
- Thanks to HPC, DNS can be performed at parameters approaching astrophysical values. This is particularly true for fingering convection, since fingers are small (typical Reynolds number moderate).

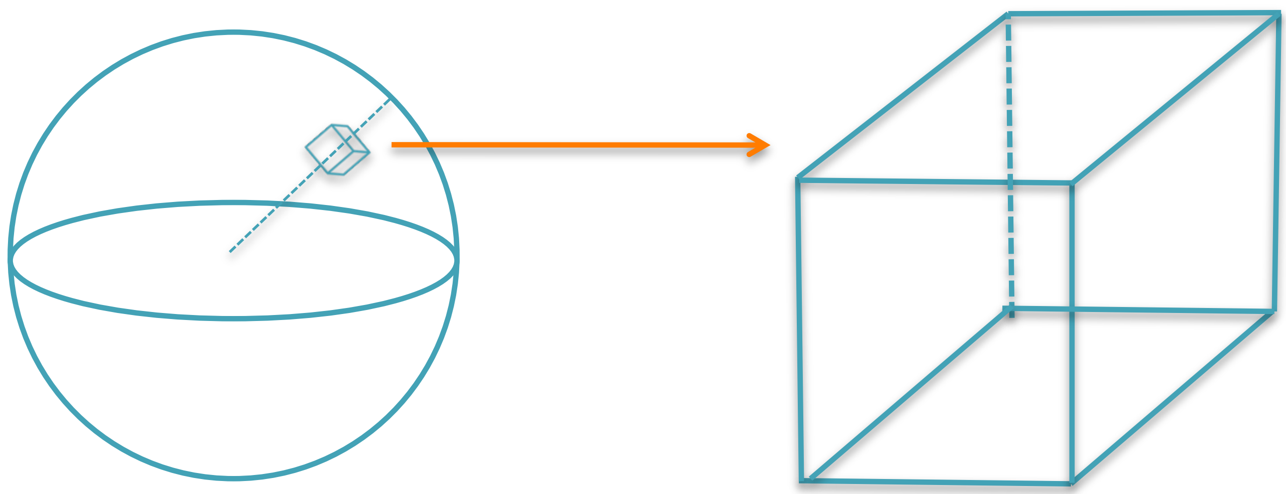
Stampede2 @ U. Texas
XSEDE facilities

Mathematical modeling

Model considered is same as before:

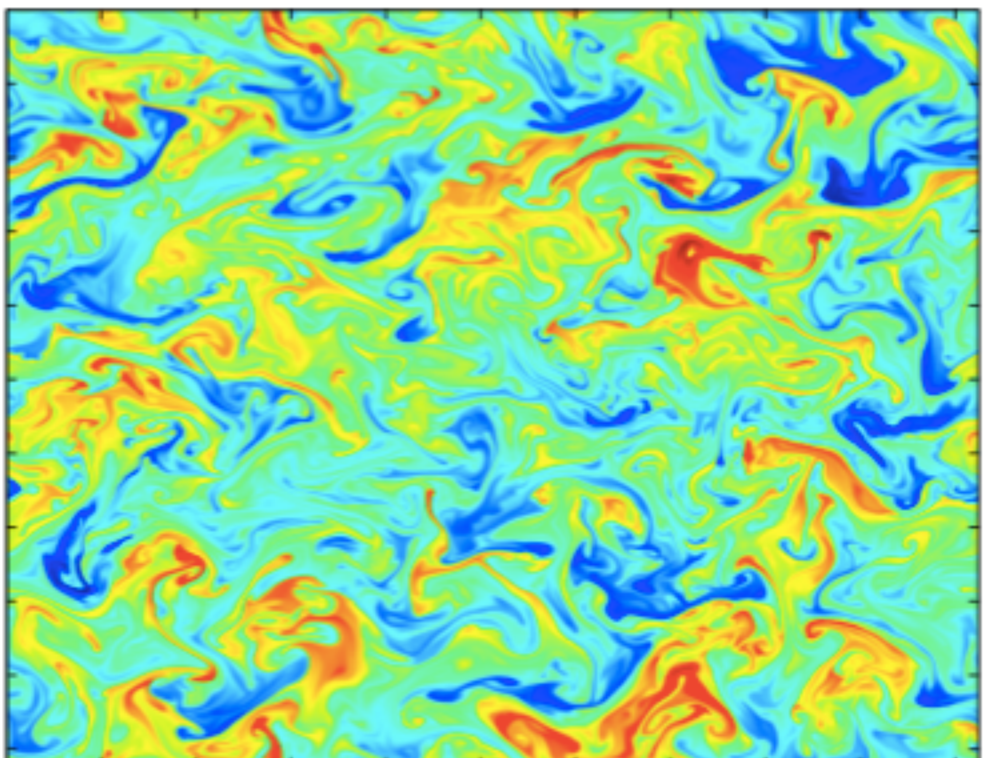
- Assume **background** temperature or salinity profiles are linear (constant gradients T_{0z}, C_{0z})
- Let $T'(x,y,z,t) = zT_{0z} + \tilde{T}(x,y,z,t)$ and $C'(x,y,z,t) = zC_{0z} + \tilde{C}(x,y,z,t)$
- Assume that all **perturbations** are triply-periodic in domain (L_x,L_y,L_z)

- This enables us to study the phenomenon with little influence from boundaries.



2D vs. 3D

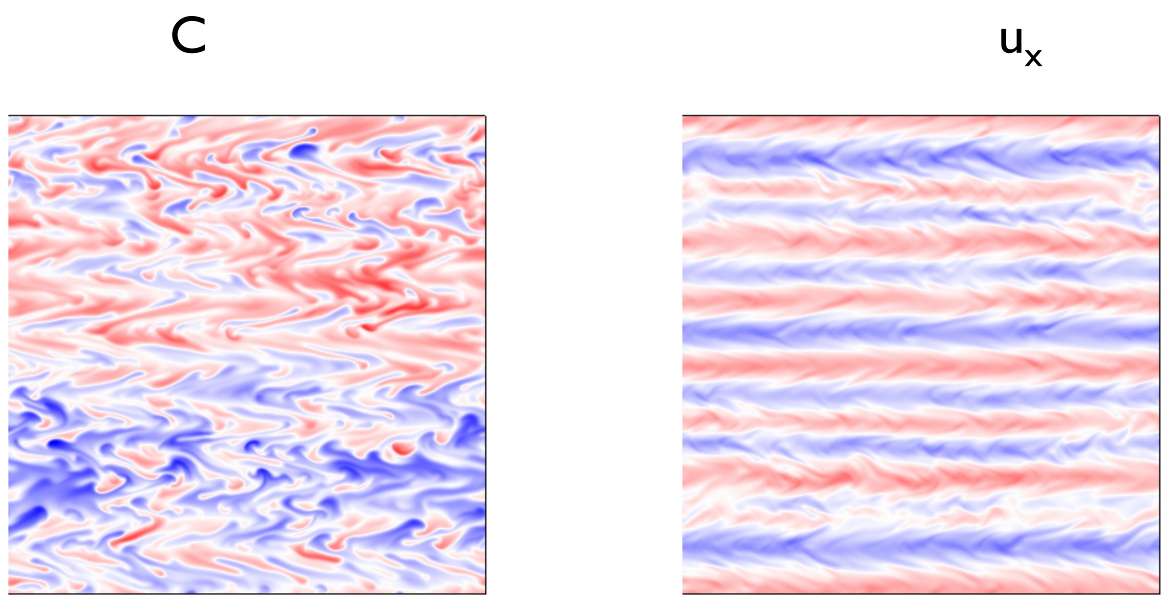
- 2D simulations are very tempting, as they are a fraction of the computational cost, and can be run on desktop computer with simple serial code.
- Early numerical work (e.g. Dennisenkov 2010) used 2D model.



Compositional field
2D fingering convection
in RGB star,
Denissenkov 2010

2D vs 3D

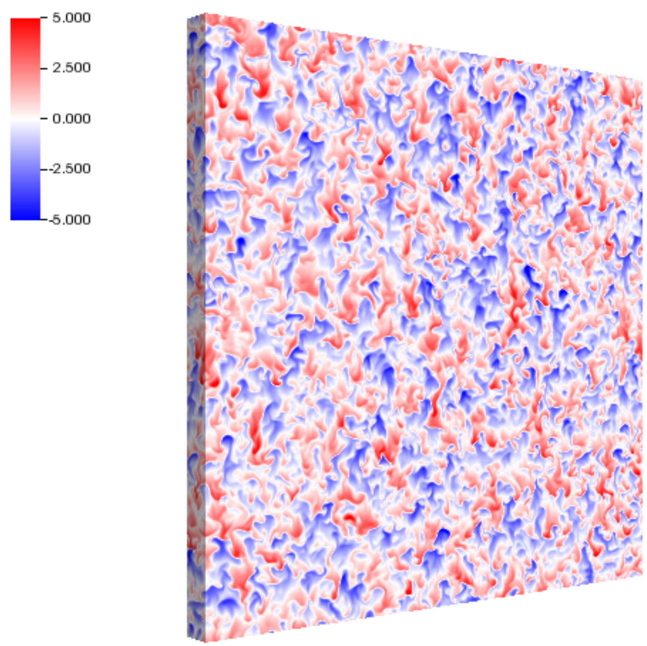
- At low Prandtl number, there is a huge difference between 2D and 3D simulations.
  - Pr = τ = 0.03, R_0=33, 2D case: artificial shear layers

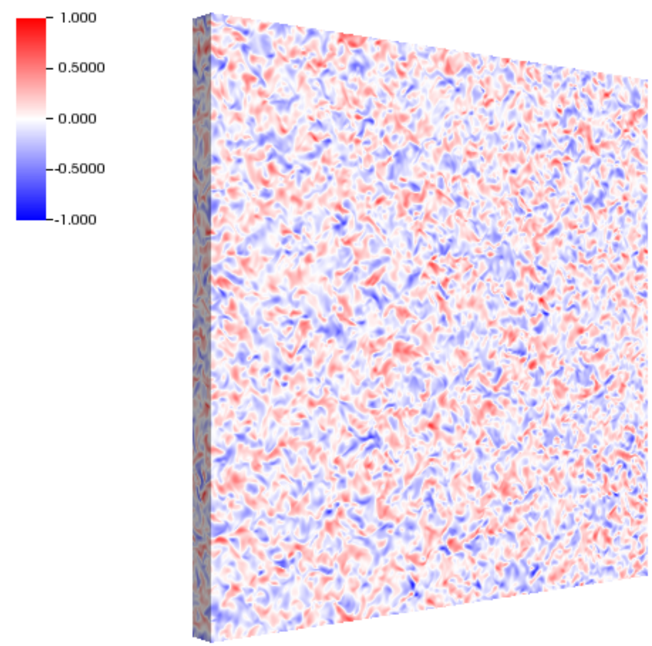


Garaud and Brummell 2015

2D vs 3D

- At low Prandtl number, there is a huge difference between 2D and 3D simulations.
  - Pr = τ = 0.03, R_0=33, 3D case (thin domain): no shear layers

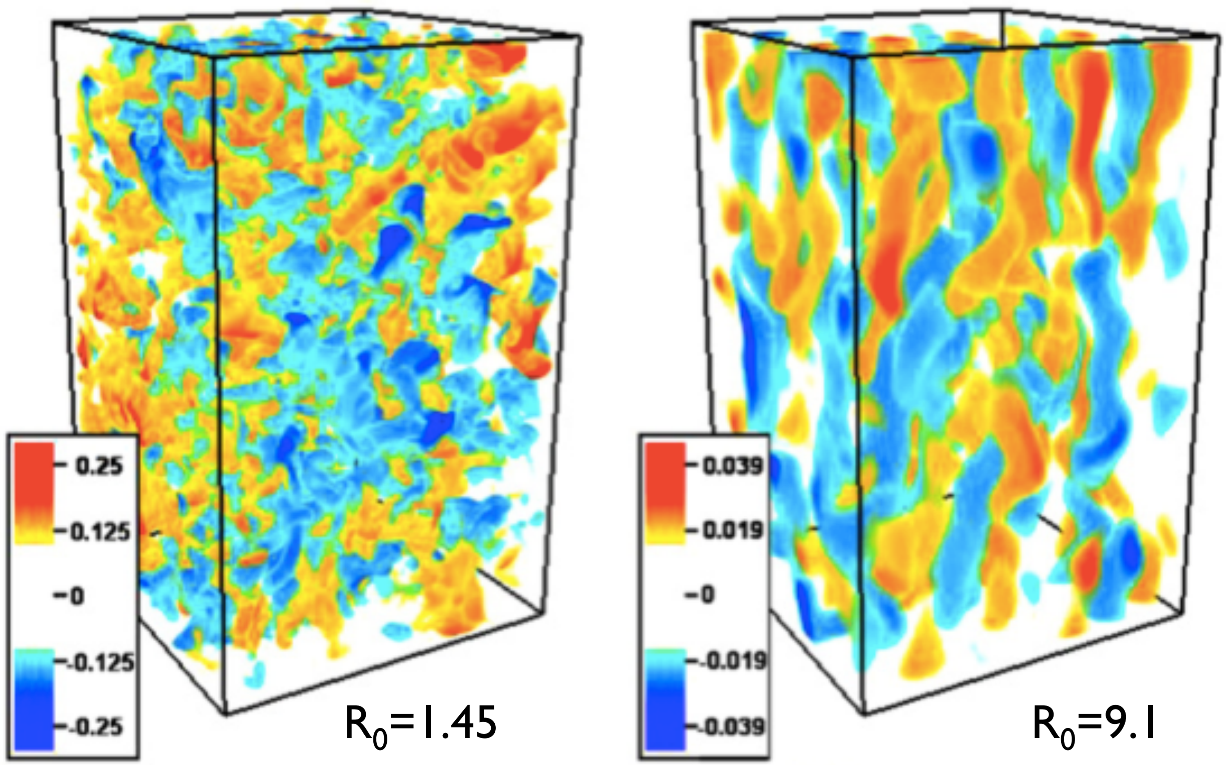




Garaud and Brummell 2015

3D simulations of fingering convection

- Early 3D work first presented by Traxler et al. 2011.

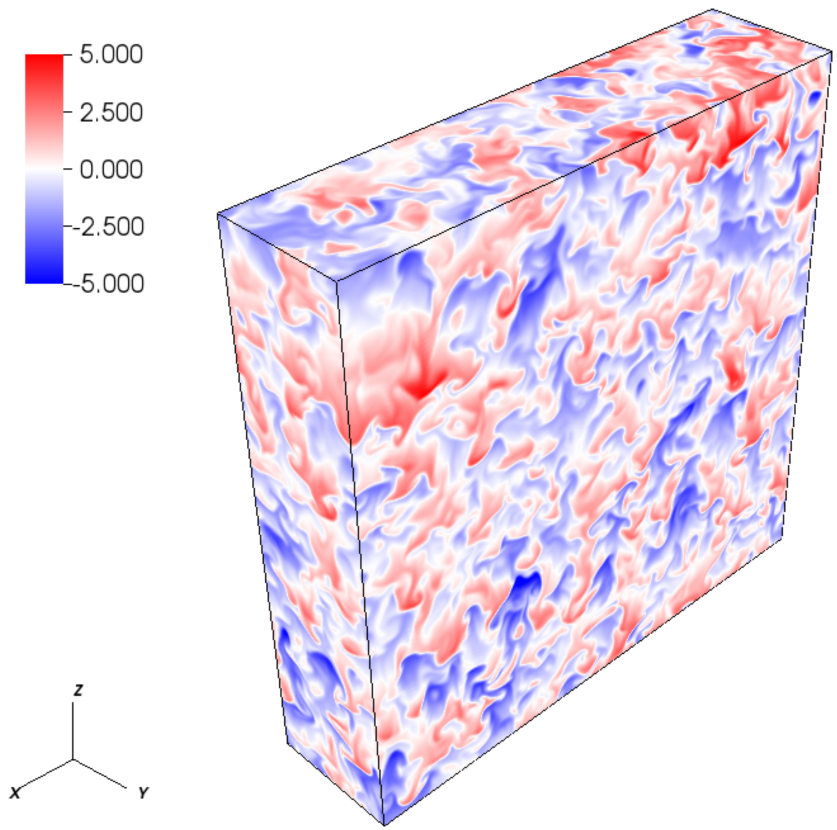


Pr = τ = 0.1

- Parameters not "astrophysical" but trying to be…

3D simulations of fingering convection

- More recent work (Brown et al. 2013) reaches smaller Pr, τ



Pr = τ =0.01,
R_0 = 5

More than just a pretty movie …

- In DNS, there is no "mixing coefficient". Compositional transport is caused by actual fluid motion, and accounted for exactly through the compositional equation (dimensional form)

$$\frac{\partial C}{\partial t} + \mathbf{u}\cdot\nabla C + wC_{0z} = \kappa_C \nabla^2 C$$

- Horizontal average of this equation becomes

$$\frac{\partial \bar{C}}{\partial t} + \frac{\partial}{\partial z}\overline{wC} = \kappa_C \frac{\partial^2 \bar{C}}{\partial z^2}$$

which can be rewritten in conservative form as $\frac{\partial \bar{C}}{\partial t} + \frac{\partial F_{C,tot}}{\partial z} = 0$

where

$$F_{C,tot} = \overline{wC} - \kappa_C \frac{\partial \bar{C}}{\partial z}$$

Turbulent (macroscopic) flux → $\overline{wC}$

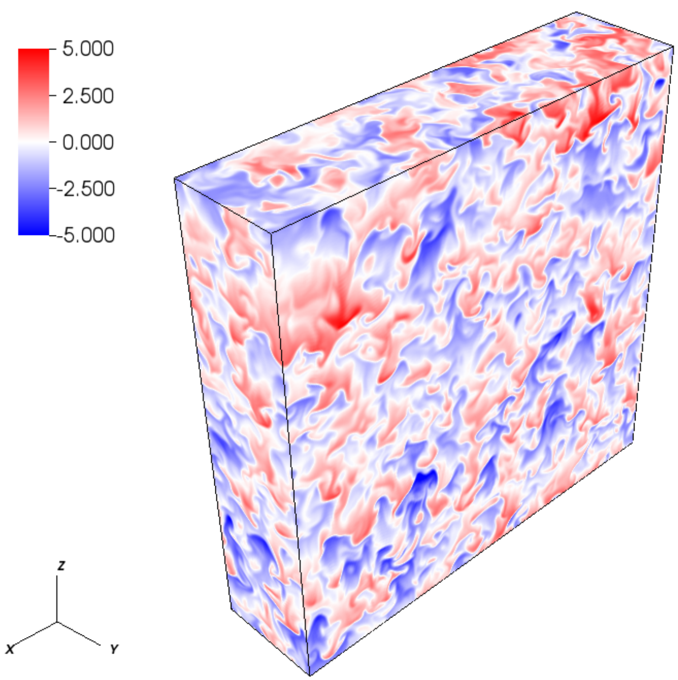
Diffusive (microscopic) flux → $\kappa_C \frac{\partial \bar{C}}{\partial z}$

- In a homogeneous steady state, the flux is constant in the domain:

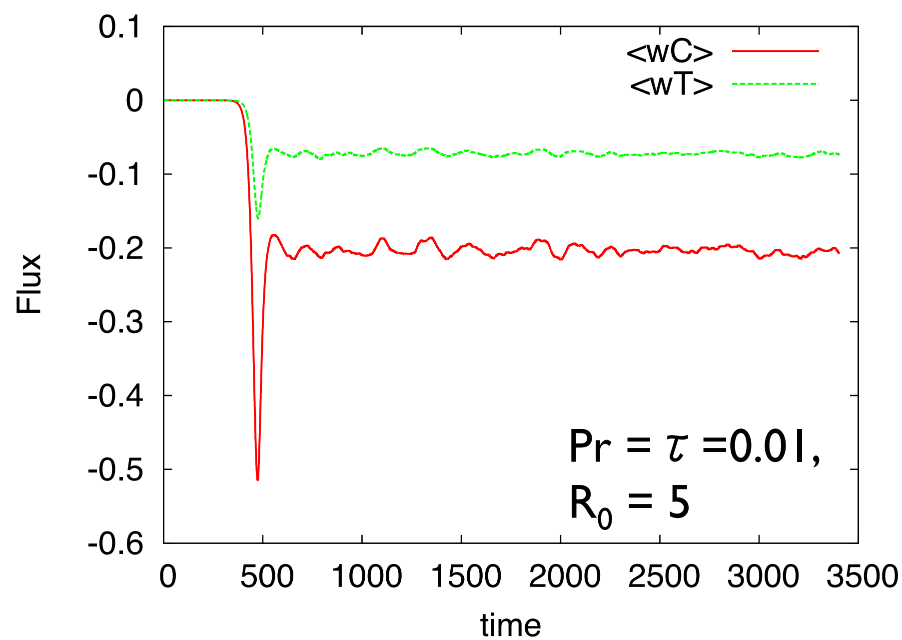
$$\overline{wC} = \langle wC \rangle$$

Modeling transport

- After exponential growth, nonlinear saturation leads to statistically stationary state.
- Both compositional and temperature flux are negative (transport is downward)



(non-dimensional)



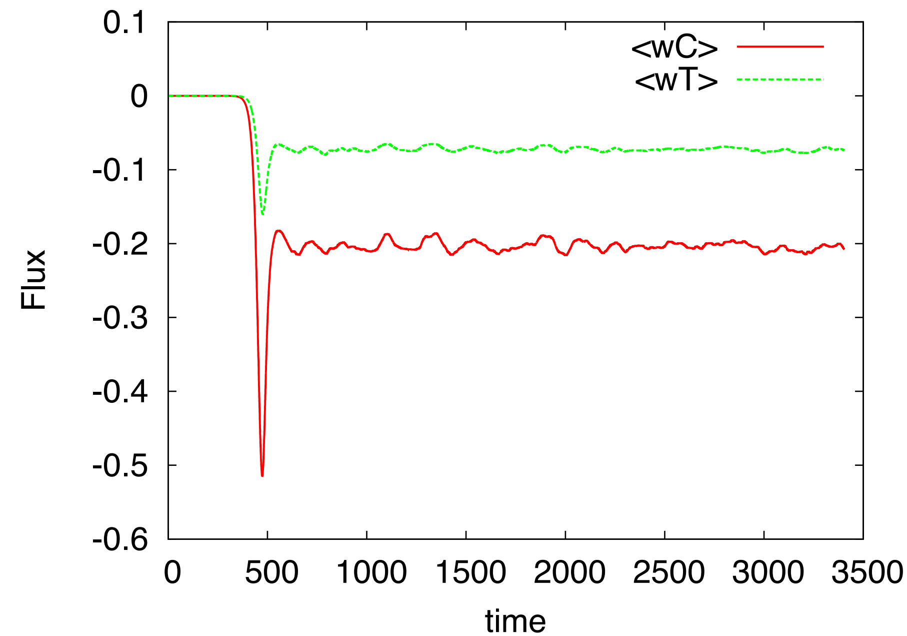
Modeling transport

- After exponential growth, nonlinear saturation leads to statistically stationary state.
- Both compositional and temperature flux are negative (transport is downward)
- This is easy to understand:

(derivation on board)

- In a star, this means that temperature is transported upgradient! (but not much)

(non-dimensional)



3D simulations of fingering convection

- To extract the turbulent diffusion coefficient, assume :

$$F_C = \langle wC \rangle = -D_{fing} \nabla C$$

- As a result, we define $D_{fing} = -\dfrac{\langle wC \rangle}{C_{0z}} \rightarrow D_{fing} = -\langle \hat{w}\hat{C} \rangle R_0 \kappa_T$

Can be directly extracted from DNS as function of input parameters Pr, τ , R_0

3D simulations of fingering convection

- To extract the turbulent diffusion coefficient, assume :

$$F_C = \langle wC \rangle = -D_{fing} \nabla C$$

- As a result, we define $D_{fing} = -\dfrac{\langle wC \rangle}{C_{0z}} \rightarrow D_{fing} = -\langle \hat{w}\hat{C} \rangle R_0 \kappa_T$

- Data will often be presented in terms of the non-dimensional **Nusselt number**

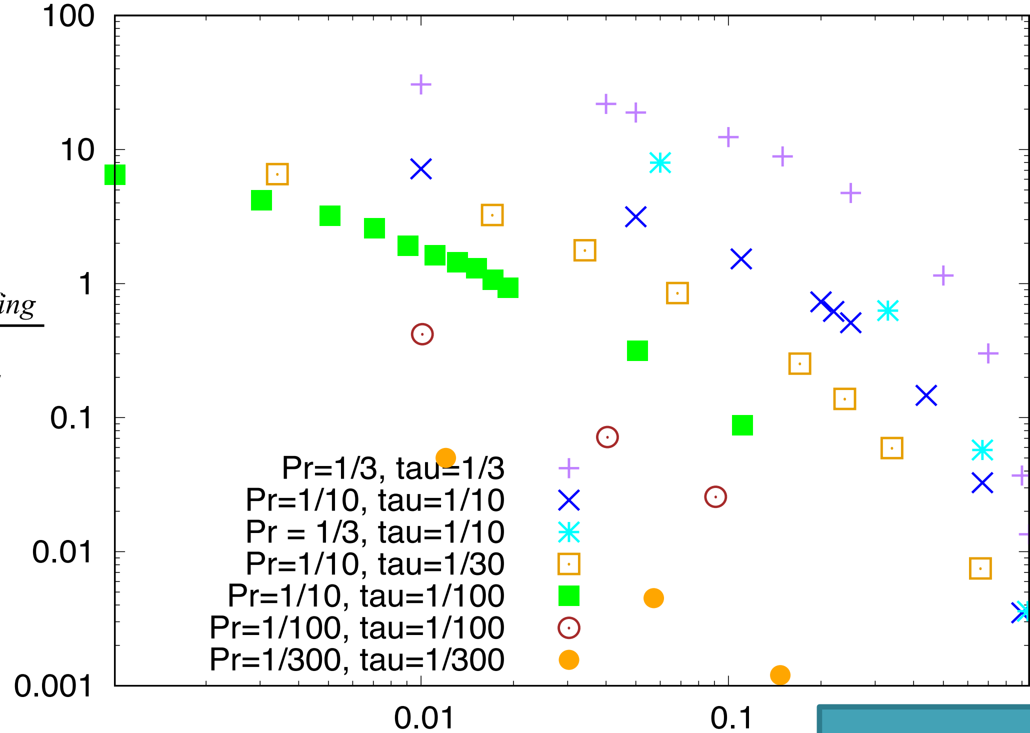
$$\mathrm{Nu}_C = \frac{-\kappa_C C_{0z} + \langle wC \rangle}{-\kappa_C C_{0z}} = \frac{\kappa_C + D_{fing}}{\kappa_C} = \frac{\text{Total flux of composition}}{\text{Diffusive flux of composition}}$$

so $\mathrm{Nu}_C - 1 = \dfrac{D_{fing}}{\kappa_C}$ measures the efficiency of turbulent mixing

3D simulations of fingering convection

The Nusselt number (or fluxes) can be extracted for a wide range of simulations. This is the most complete dataset to date.

$$\mathrm{Nu}_T - 1 = \frac{D_{T,fing}}{\kappa_T}$$



$$r = \frac{R_0 - 1}{\tau^{-1} - 1}$$

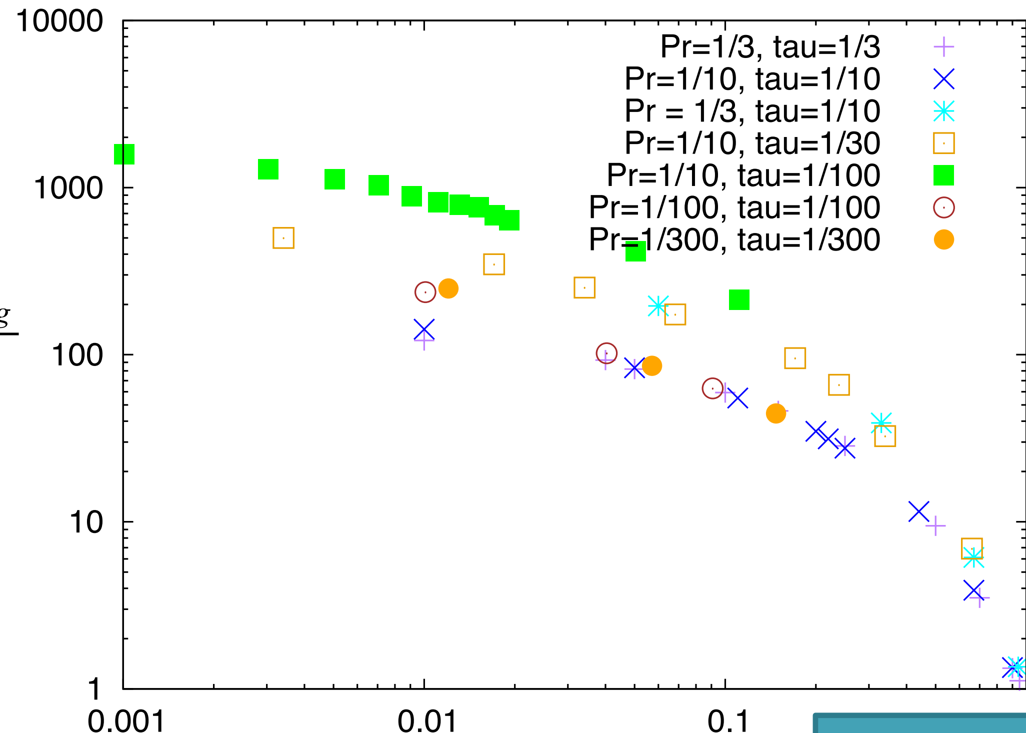
Data from Traxler et al. 2011; Radko & Smiths 2012; Brown et al. 2013; Garaud 2018

Transport of heat by fingering convection is negligible for Pr, τ <0.01

3D simulations of fingering convection

The Nusselt number (or fluxes) can be extracted for a wide range of simulations. This is the most complete dataset to date.

$$\mathrm{Nu}_C = 1 + \frac{D_{fing}}{\kappa_C}$$



$$r = \frac{R_0 - 1}{\tau^{-1} - 1}$$

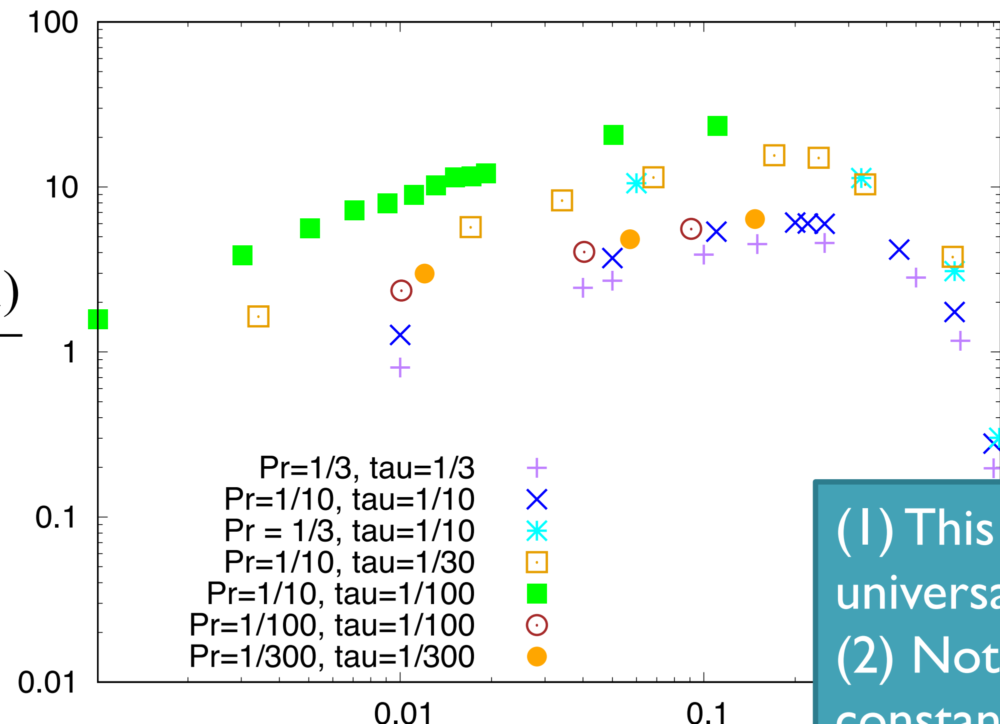
Data from Traxler et al. 2011; Radko & Smiths 2012; Brown et al. 2013; Garaud 2018

Transport of composition is much more significant.

3D simulations of fingering convection

We can use it to test the Ulrich (1972) or Kippenhahn et al. (1980) models:

$$\frac{D_{fing}(R_0 - 1)}{\kappa_T}$$



$$r = \frac{R_0 - 1}{\tau^{-1} - 1}$$

Data from Traxler et al. 2011; Radko & Smiths 2012; Brown et al. 2013; Garaud 2018

(1) This should be a universal constant!
(2) Note how Ulrich's constant is ruled out by simulations!

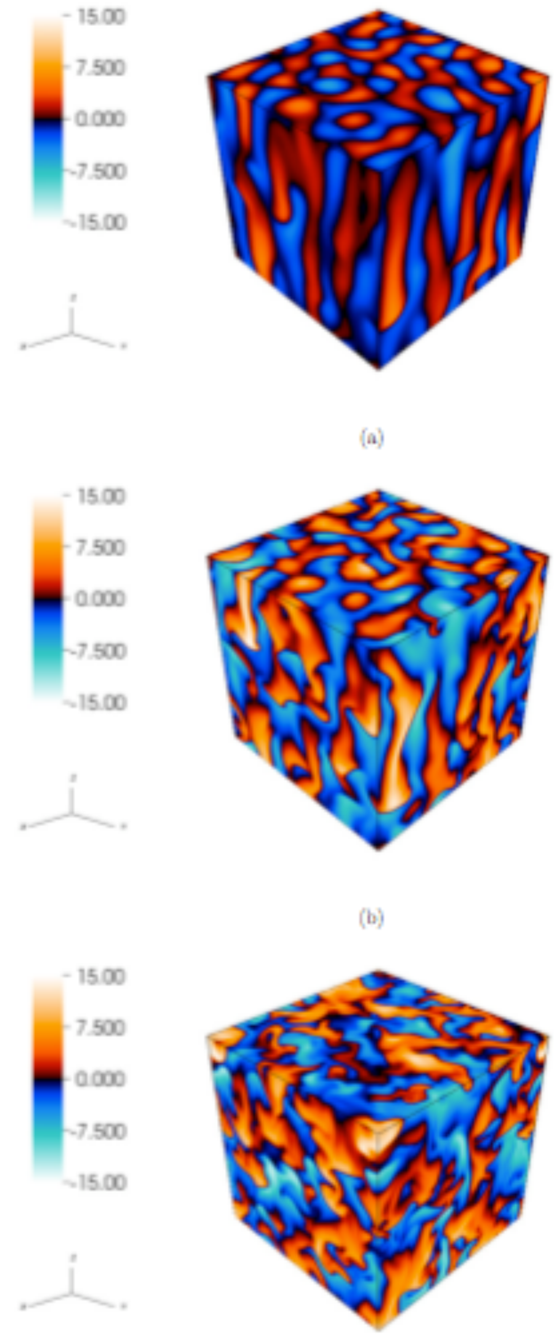
An improved model for fingering convection

Brown et al. 2013

In fingering convection in astrophysics (Brown et al. 2014 , following Radko & Smith 2012)

- Saturation occurs when the growth rate σ of the shearing instability associated with the fluid motion within the fingers is of the order of the growth rate of the fingering instability λ :

$$\sigma = K\lambda$$



Brown et al. 2013

- The growth rate λ and most-rapidly growing mode of the fingering instability can be found by solving linear problem.

- Ignoring viscosity, by dimensional analysis, (or more rigorously through Floquet theory), it can be shown that the shearing instability growth rate is

$$\sigma \propto W k_h$$

  - W = typical vertical velocity within the fingers
  - k_h = horizontal wavenumber of most rapidly-growing finger.

- So we can estimate W at saturation of the fingering instability:

$$\sigma \propto W k_h = K\lambda \rightarrow W = \frac{C_B \lambda}{k_h}$$

Brown et al. 2013

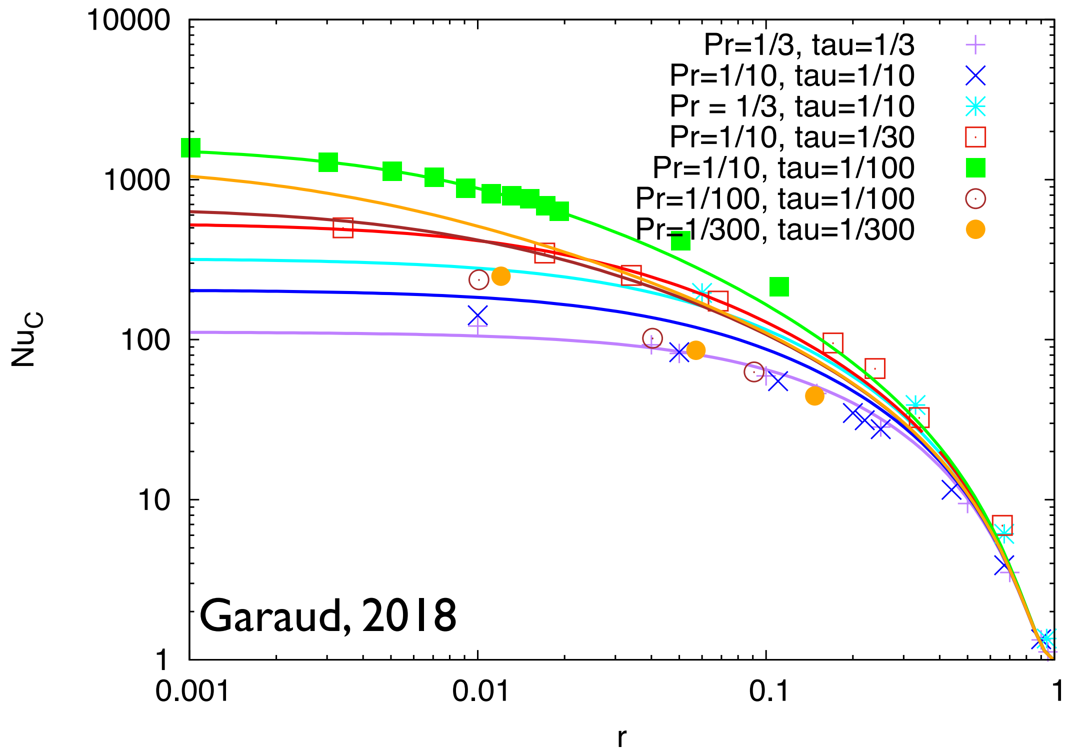
- To compute the diffusion coefficient, we then use linear theory (derivation on the board).

$$\hat{F}_C = \langle \hat{w}\hat{C} \rangle \propto -\frac{1}{R_0}\frac{W^2}{\lambda + \tau k_h^2} = -\frac{C_B^2 \lambda^2}{R_0 k_h^2 (\lambda + \tau k_h^2)}$$

Everything except C_B is known from linear theory!

Brown et al. 2013

With C_B =7, the fit is very good (within factor of 2) for all (astrophysically-relevant) cases with τ <Pr<<1

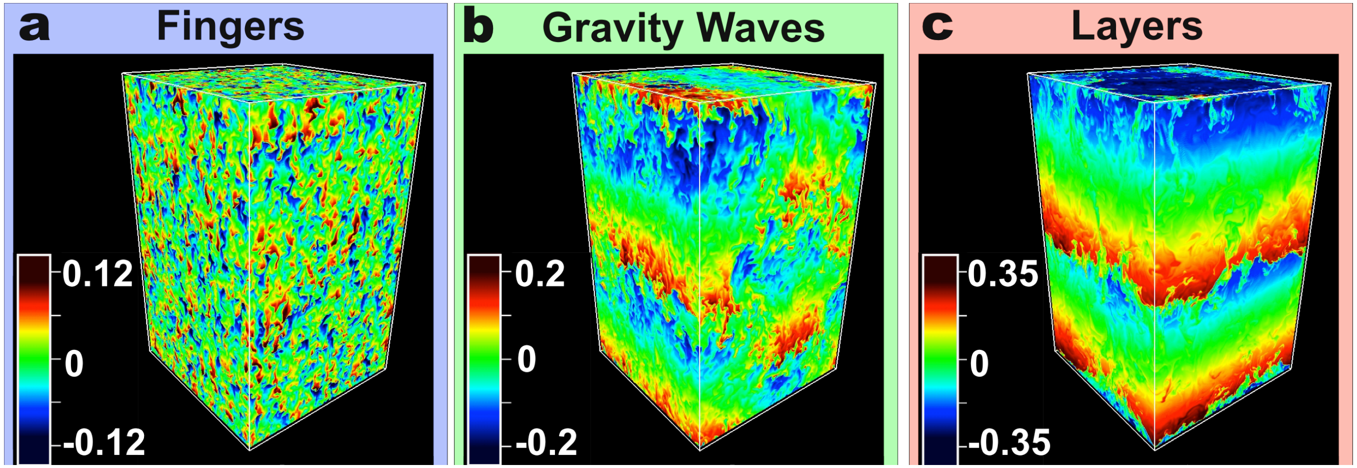


We now have a way of estimating transport by homogeneous fingering convection at astrophysical parameters.

But is it the whole story ???

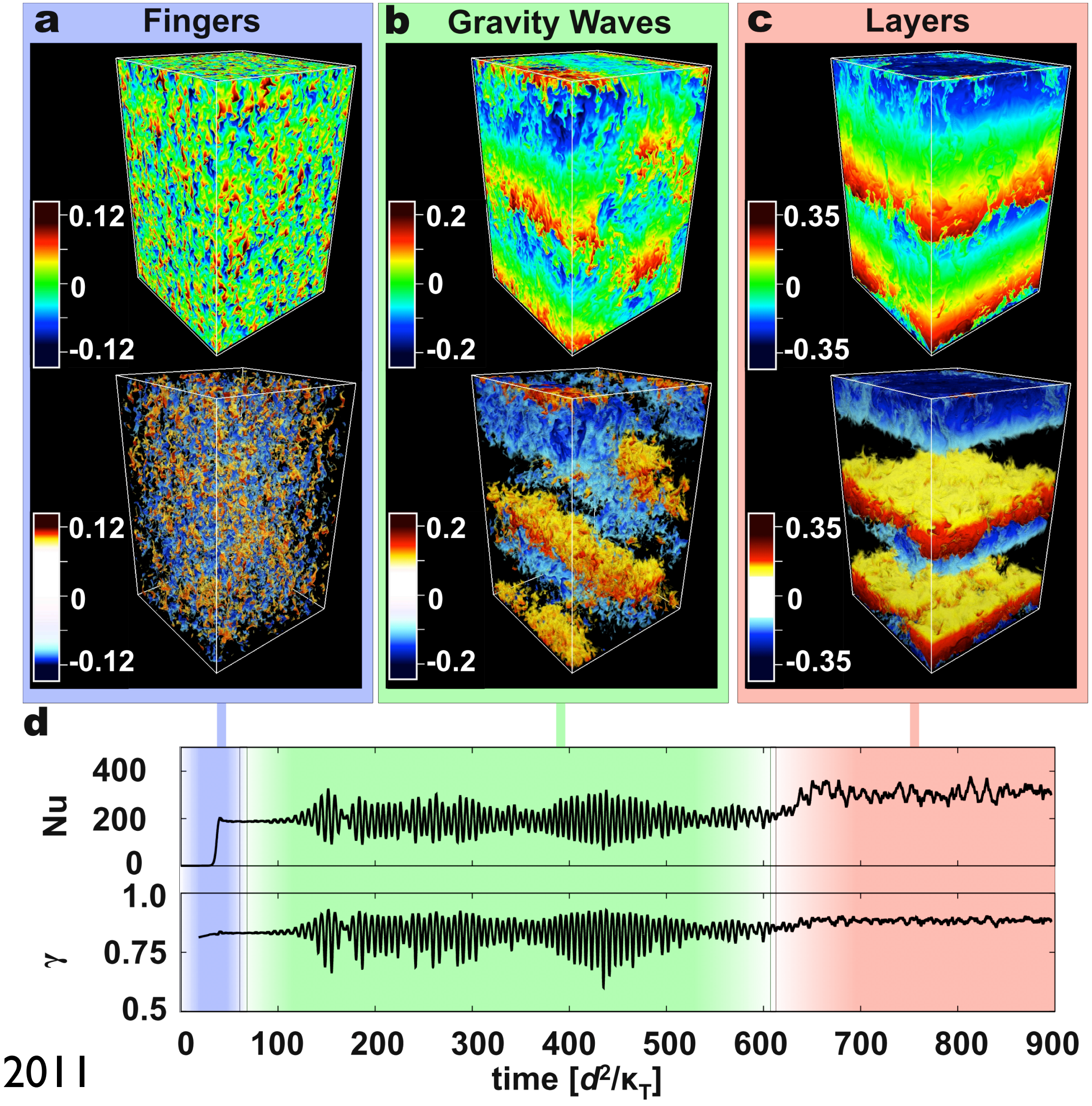
Large-scale structures

Evolution of a fingering simulation at $\mathrm{Pr} = 7,\ \tau = 1/3,\ R_0 = 1.1$



Stellmach et al. 2011

Large-scale structures

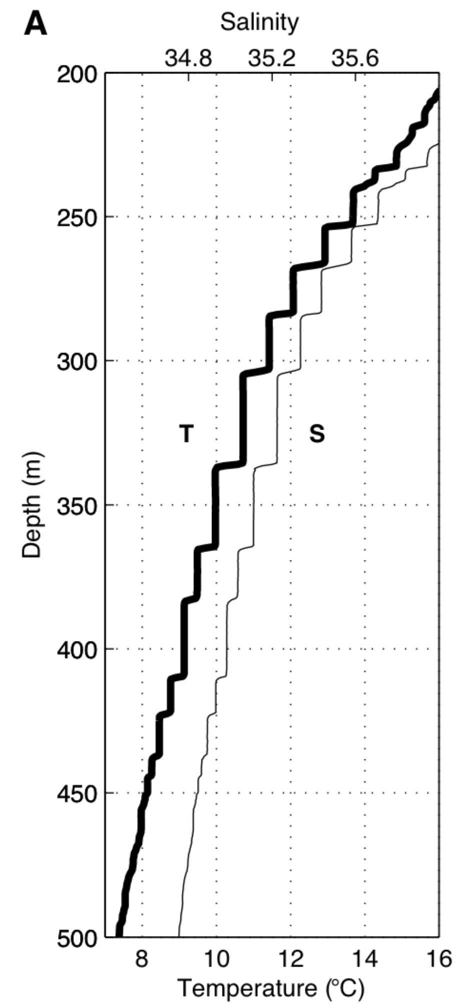


Fluxes increase significantly when layers form.

Stellmach et al. 2011

Thermohaline staircases

- In fact, the propensity of fingering convection to form layers has been known for a long time.
- In many regions of the ocean unstable to fingering convection, one can find thermohaline staircases.



Schmitt, 2005

Mean-field theory

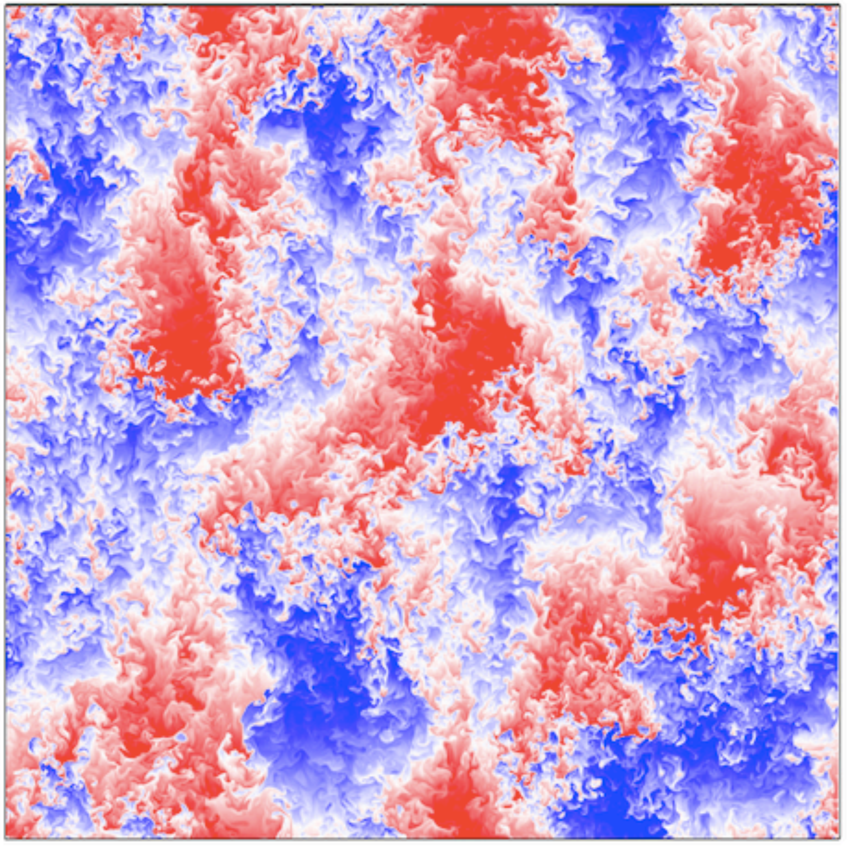
Questions:

1. Why do large-scale structures emerge?
2. Under which conditions do they emerge?
3. How to they modify transport properties?

- Large-scale structures (waves, staircases) in double-diffusive convection can be studied using "mean-field" theory

Mean-field theory

- **General idea:** large-scale structures form through positive feedback between large-scale temperature/composition perturbation and induced fluxes.



Mean-field theory

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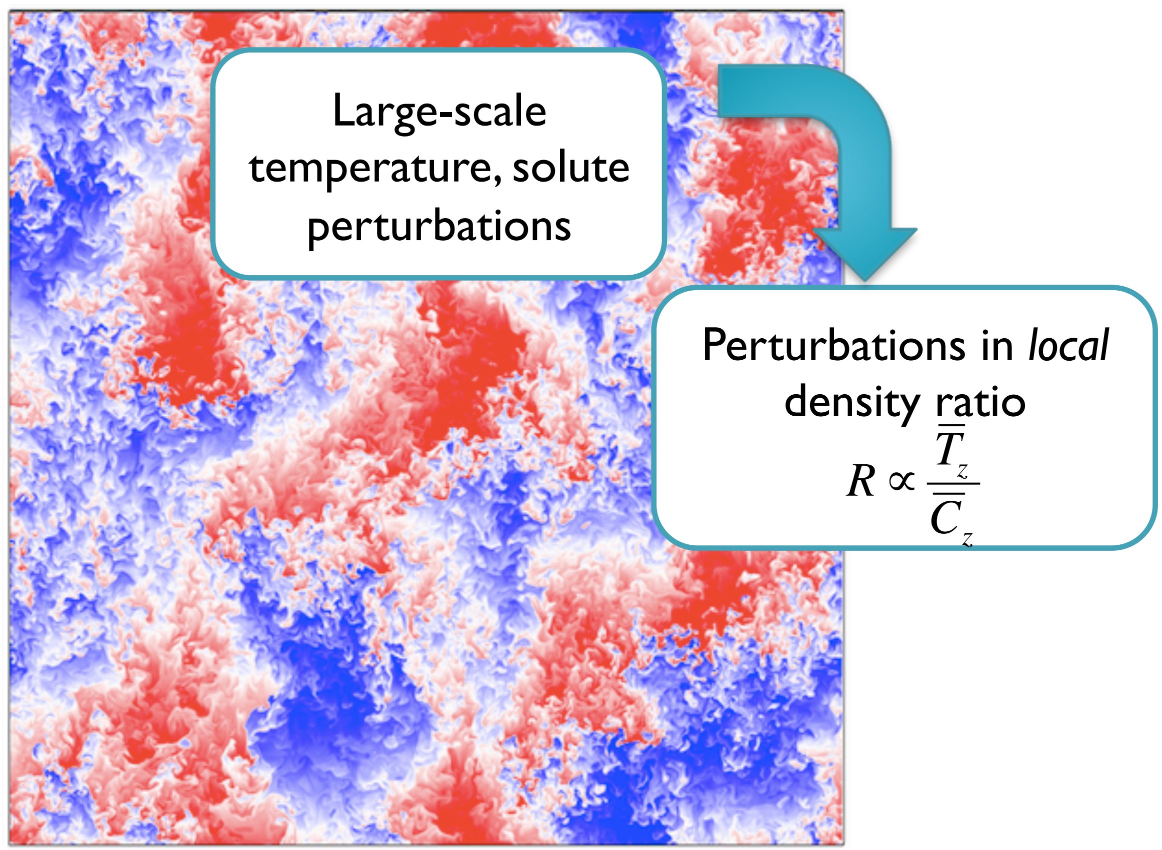

Mean-field theory

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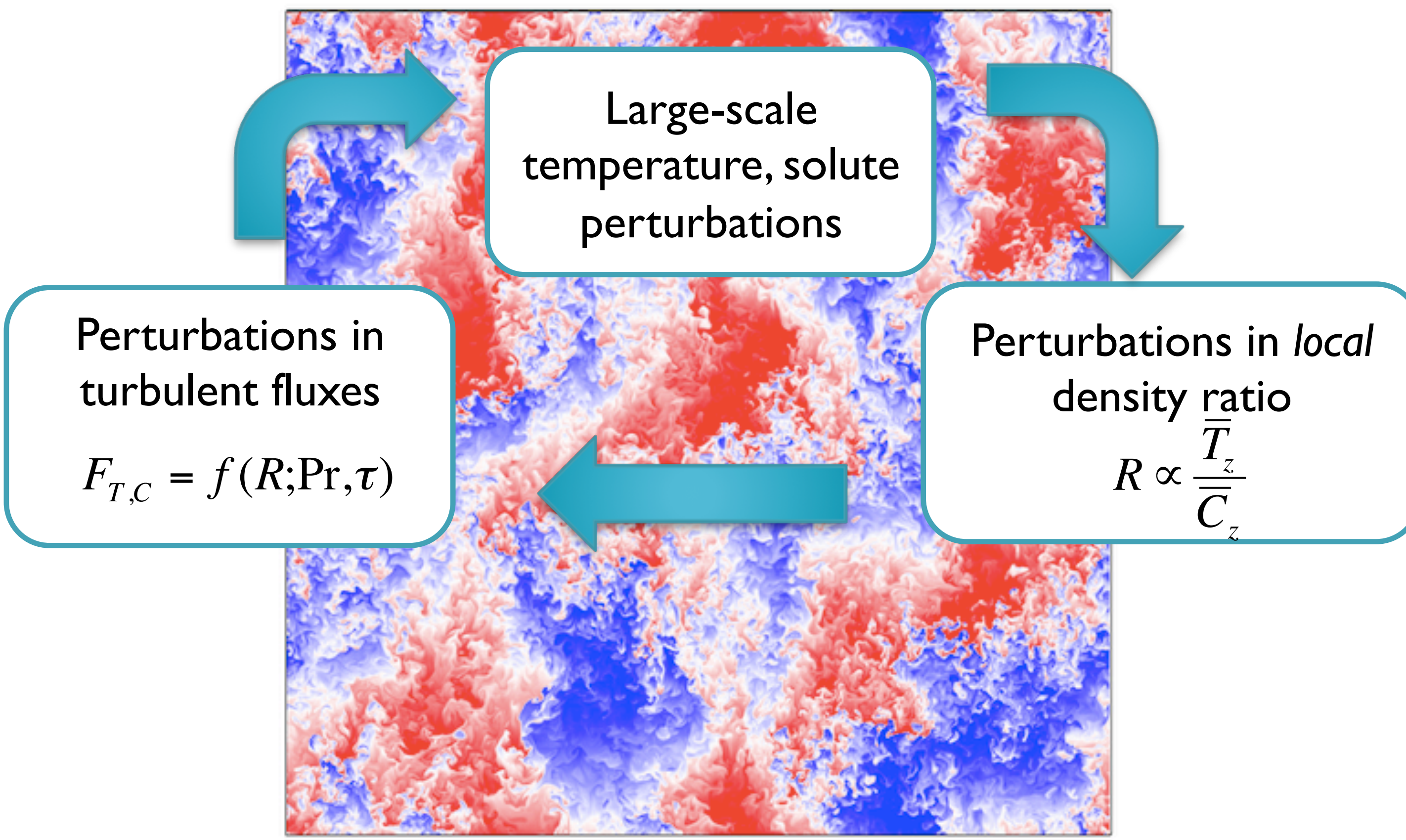
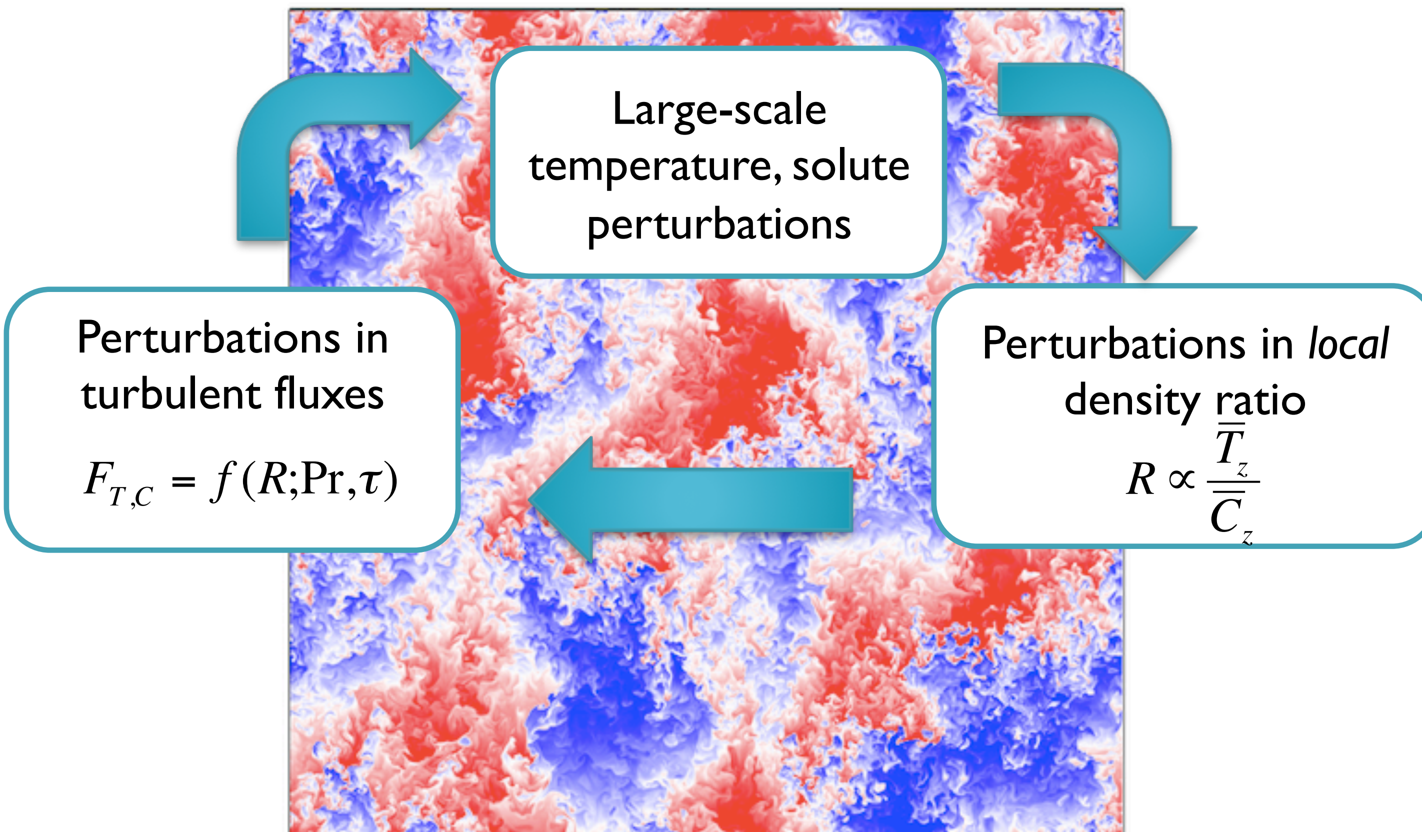


Mean-field theory

Different feedback loops can lead to different "mean-field" instabilities:

1. Gravity-wave generation
2. Layer formation



Mean-field theory

- Consider the original equations, and average them over smaller scales and fast timescales (all equations now non-dimensional)

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot \mathbf{R}\right) = -\nabla \bar{p} + \left(\bar{T} - \bar{C}\right)\mathbf{e_z} + \nabla^2 \bar{\mathbf{u}}$$

$$\frac{\partial \bar{T}}{\partial t} + \nabla \cdot \mathbf{F}_T + \bar{w} = \nabla^2 \bar{T}$$

$$\rightarrow \quad \frac{\partial \bar{T}}{\partial t} + \bar{w} = -\nabla \cdot \mathbf{F}_{T,\mathrm{tot}}$$

$$\frac{\partial \bar{C}}{\partial t} + \nabla \cdot \mathbf{F}_C + \frac{\bar{w}}{R_0} = \tau \nabla^2 \bar{C}$$

$$\rightarrow \quad \frac{\partial \bar{C}}{\partial t} + \frac{\bar{w}}{R_0} = -\nabla \cdot \mathbf{F}_{C,\mathrm{tot}}$$

$$\mathbf{F}_{T,\mathrm{tot}} = -\nabla \bar{T} + \overline{\mathbf{u}T}$$

$$\mathbf{F}_{C,\mathrm{tot}} = -\tau \nabla \bar{C} + \overline{\mathbf{u}C}$$

Note: here the overbar denotes averaging process, which needs not be horizontal average

- Standard mean field closure problem: if Reynolds stresses and fluxes are known, the problem can be solved for evolution of large-scale fields.

Mean-field theory

Empirical "closure" model: (Radko 2003; Traxler et al. 2011)

1. Neglect Reynolds stress.
2. Assume fluxes are mostly in vertical direction, and define non-dimensional quantities

$$\mathrm{Nu}_T = \frac{F_{T,\mathrm{tot}}}{-(1+\bar{T}_z)} \quad \text{and} \quad \gamma = \frac{F_{T,\mathrm{tot}}}{F_{C,\mathrm{tot}}}$$

3. Assume these non-dimensional quantities only depend on other non-dimensional quantities

$$\mathrm{Nu}_T = \mathrm{Nu}_T(R;\mathrm{Pr},\tau) \quad \text{and} \quad \gamma = \gamma(R;\mathrm{Pr},\tau)$$

where

$$R = \frac{1+\bar{T}_z}{R_0^{-1}+\bar{C}_z}$$

is the *local* density ratio, and the functions Nu_T and γ are assumed to be known (see later about this).

Mean-field theory

- Mean field equations for staircase formation boil down to:

$$\begin{cases} \frac{1}{\Pr}\frac{\partial \bar{\mathbf{u}}}{\partial t} = -\nabla \bar{p} + (\bar{T} - \bar{C})\mathbf{e}_z + \nabla^2 \bar{\mathbf{u}} \\ \frac{\partial \bar{T}}{\partial t} + \bar{w} = -\nabla \cdot F_{T,\mathrm{tot}} \\ \frac{\partial \bar{C}}{\partial t} + \frac{\bar{w}}{R_0} = -\nabla \cdot F_{C,\mathrm{tot}} \end{cases}$$

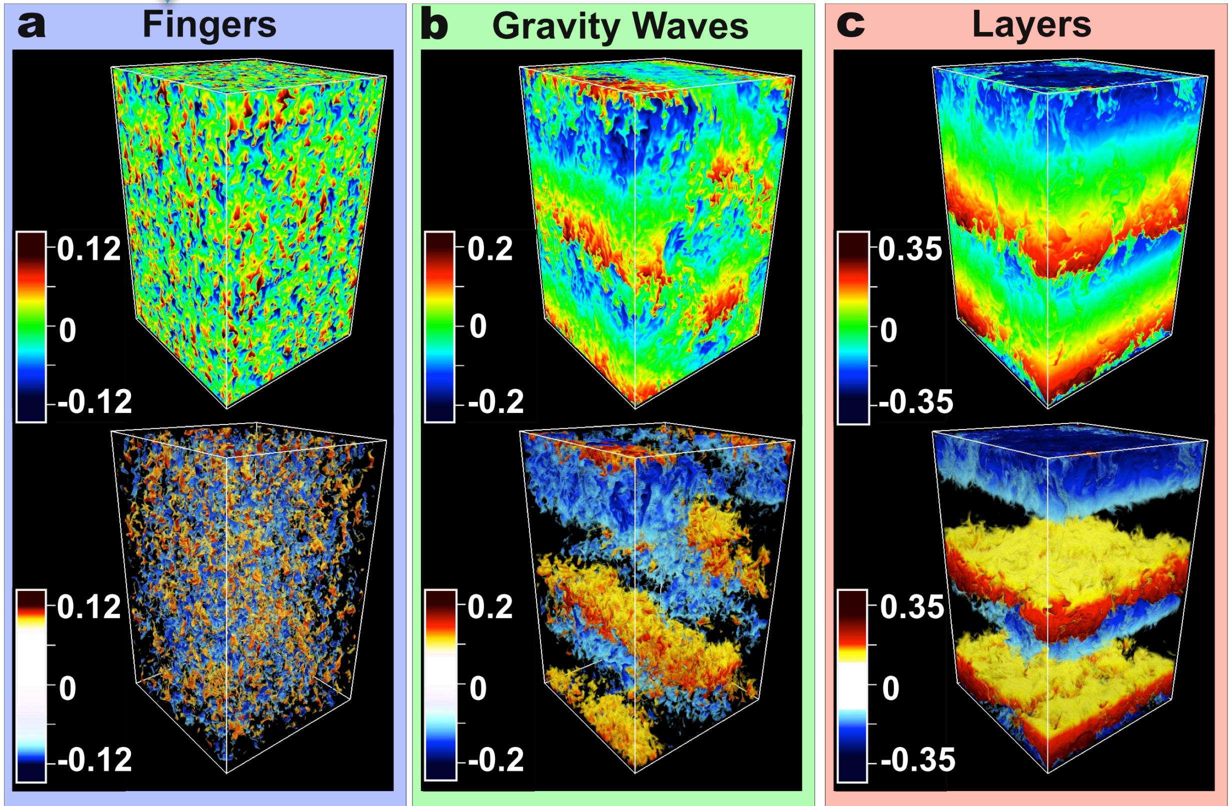
$$\begin{cases} F_{T,\mathrm{tot}} = -\mathrm{Nu}_T\left(1 + \bar{T}_z\right) \quad \text{and} \quad F_{C,\mathrm{tot}} = \frac{F_{T,\mathrm{tot}}}{\gamma} \\ \mathrm{Nu}_T = \mathrm{Nu}_T(R;\Pr,\tau) \\ \gamma = \gamma(R;\Pr,\tau) \\ R = \frac{1+\bar{T}_z}{R_0^{-1} + \bar{C}_z} \end{cases}$$

A closed set of nasty coupled nonlinear equations!

- These equations nevertheless admit one set of simple solutions:
  - No mean flow: $\bar{\mathbf{u}} = 0$
  - Constant temperature and compositional gradients: $\bar{T}_z = T_{0z}$, $\bar{C}_z = C_{0z}$
  - Constant density ratio $R = R_0$
- This solution represents the *homogeneous* fingering state.

Mean field theory

Homogeneous fingering state.



Stellmach et al. 2011

Mean-field theory

- Let's linearize the system around the homogeneous fingering state, and study the effect of large-scale/slow timescale small amplitude perturbations.

$$\bar{T} = z + \bar{T}' \text{ and } \bar{C} = zR_0^{-1} + \bar{C}'$$

$$R(\mathbf{x},t) = R_0 + R'(\mathbf{x},t) \cong R_0 + R_0(\bar{T}'_z - R_0\bar{C}'_z)$$

$$\mathrm{Nu}_T = \mathrm{Nu}_T(R_0) + R'\left.\frac{d\mathrm{Nu}_T}{dR}\right|_{R=R_0} \quad \text{and} \quad \gamma = \gamma(R_0) + R'\left.\frac{d\gamma}{dR}\right|_{R=R_0}$$

- Substituting this back into the governing equations, to get a linearized system for large-scale variables.

Mean-field theory

- For horizontally-invariant perturbations, this is quite easy to do: equations reduce to

$$\frac{\partial \bar{T}}{\partial t} = -\nabla \cdot F_{T,\mathrm{tot}} \qquad F_{T,\mathrm{tot}} = -\mathrm{Nu}_T\left(1+\bar{T}_z\right) \quad \text{and} \quad F_{C,\mathrm{tot}} = \frac{F_{T,\mathrm{tot}}}{\gamma}$$

$$\mathrm{Nu}_T = \mathrm{Nu}_T(R;\mathrm{Pr},\tau)$$

$$\frac{\partial \bar{C}}{\partial t} = -\nabla \cdot F_{C,\mathrm{tot}} \qquad \gamma = \gamma(R;\mathrm{Pr},\tau)$$

$$R = \frac{1+\bar{T}_z}{R_0^{-1}+\bar{C}_z}$$

- Assuming normal modes of the kind $q(z,t) \propto e^{iKz+\Lambda t}$

- Get a quadratic for growth rate of horizontally-invariant modes.

$$\Lambda^2 + \Lambda k^2\left[A_{\mathrm{Nu}}(1-R_0\gamma_0^{-1}) + \mathrm{Nu}_0(1-A_\gamma R_0)\right] - k^4 A_\gamma \mathrm{Nu}_0^2 R_0 = 0$$

(derivation on the board)

Mean-field theory

- Assuming normal modes of the kind $q(\mathbf{x},t) \propto e^{i\mathbf{k}\cdot\mathbf{x}+\Lambda t}$

yields a cubic (again) for the growth rate of *large-scale* structures:

$$\Lambda^3 + a\Lambda^2 + b\Lambda + c = 0$$

where the coefficients are functions of $(\mathrm{Pr},\tau,R_0,\mathbf{k})$ *as well as*

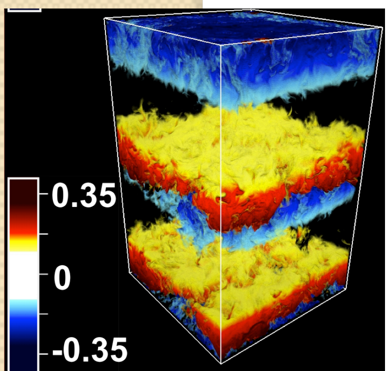
$$\mathrm{Nu}_0 = \mathrm{Nu}(R_0) \qquad \gamma_0 = \gamma_{tot}(R_0)$$

$$A_{\mathrm{Nu}} = R_0 \left.\frac{d\mathrm{Nu}}{dR}\right|_{R=R_0} \qquad A_\gamma = R_0 \left.\frac{d\gamma_{tot}^{-1}}{dR}\right|_{R=R_0}$$

- This cubic can have direct modes, or complex-conjugate modes.

Mean-field theory

- **Modes of instability:**
  - "Layering mode" or "γ-mode" (Radko 2003). The fastest growing mode is *horizontally invariant* with no mean flow.



Radko's γ-instability criterion: A necessary condition for the layering instability is that the flux ratio should be a decreasing function of density ratio: $\frac{d\gamma}{dR} < 0$

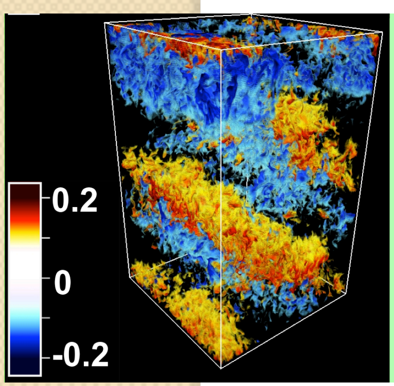
Interpretation: The horizontally invariant mean-field equations can be re-written as

$$\frac{\partial R}{\partial t} = \mathrm{Nu}_T \frac{d\gamma}{dR} \frac{\partial^2 R}{\partial z^2} + \ldots$$

➔ If $\frac{d\gamma}{dR} < 0$ then the system is antidiffusive!

Mean-field theory

- **Modes of instability:**
  - CC-modes: Large-scale exponentially growing gravity waves, and correspond to the "collective instability" (Stern 1969).



Interpretation: The collective modes are simply the ODDC instability using turbulent diffusivities!

- from a turbulent point of view, the salt diffuses faster than heat
- now the rapidly-diffusive component is unstably stratified, while the slowly diffusing one is stable.

Criterion for instability: Turbulent diffusivities must be sufficiently large.

Mean-field theory: proof of concept

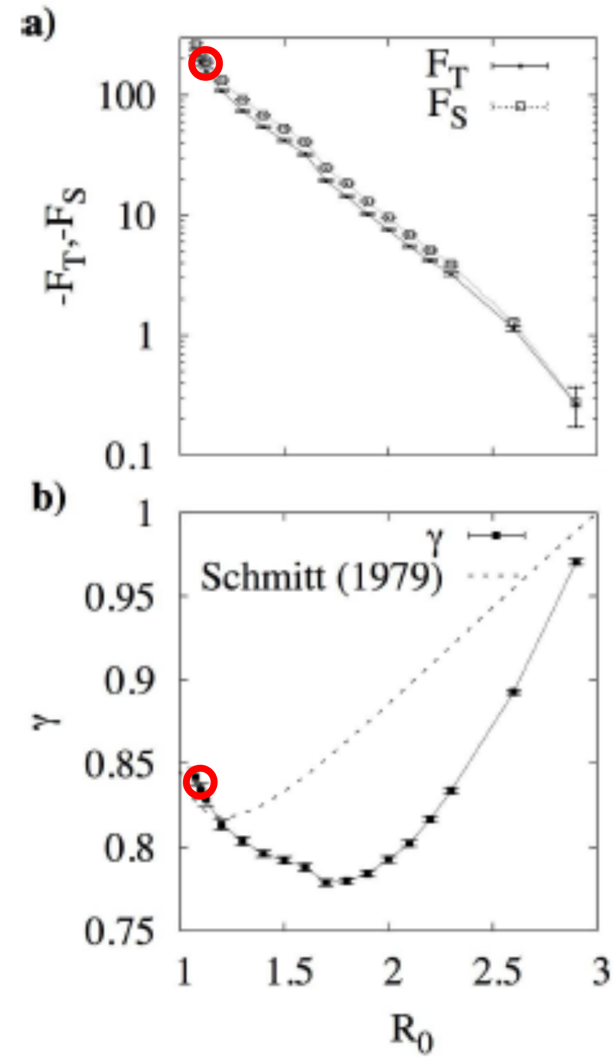
To determine whether this works quantitatively

- We need to find out what are the functions

$$\mathrm{Nu}_T = \mathrm{Nu}_T(R;\mathrm{Pr},\tau)$$

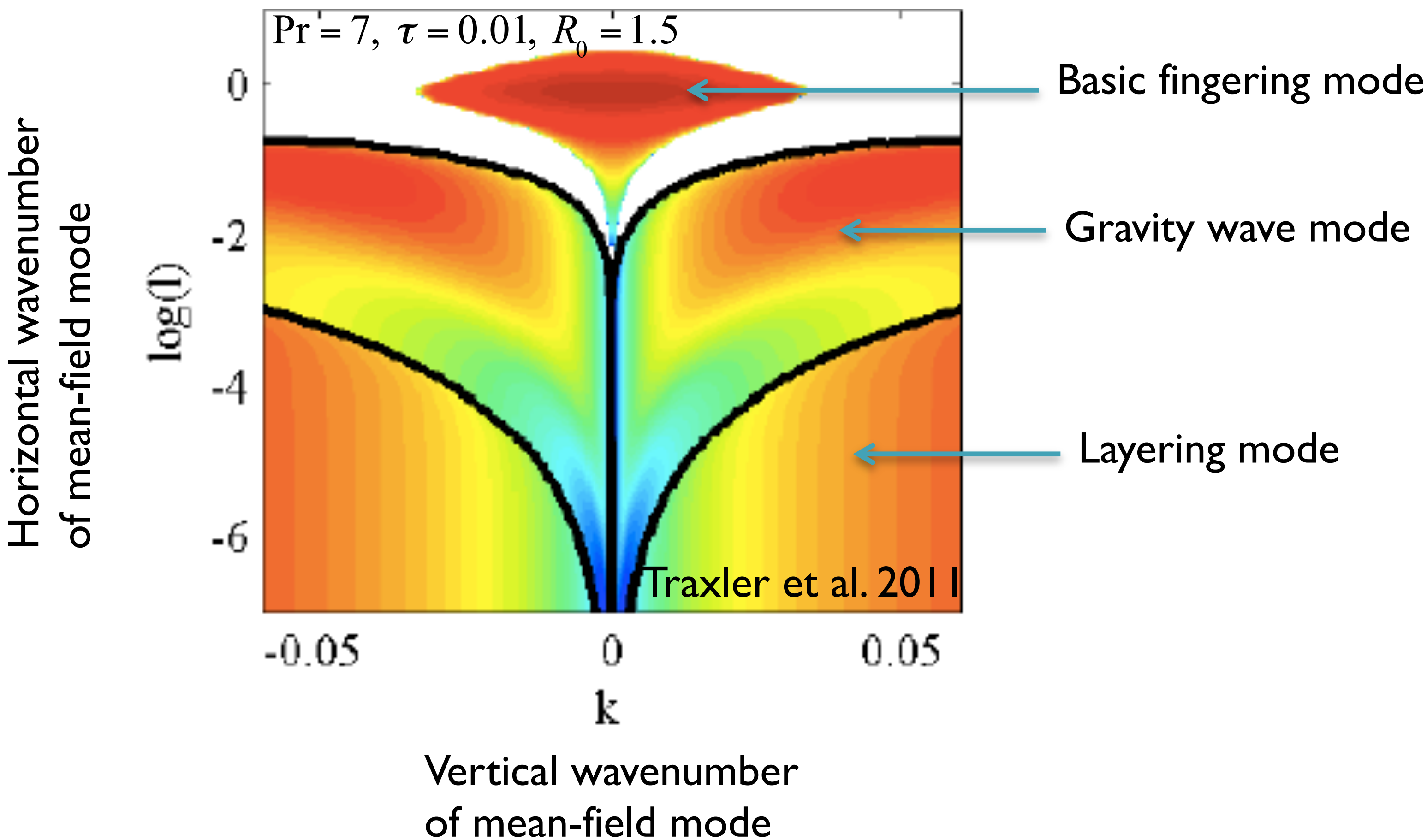
$$\gamma = \gamma(R;\mathrm{Pr},\tau)$$

- For water parameters, we used small-box simulations to extract these quantities (Traxler et al. 2011; Stellmach et al. 2011) and their derivatives with respect to R.



Mean-field theory: proof of concept

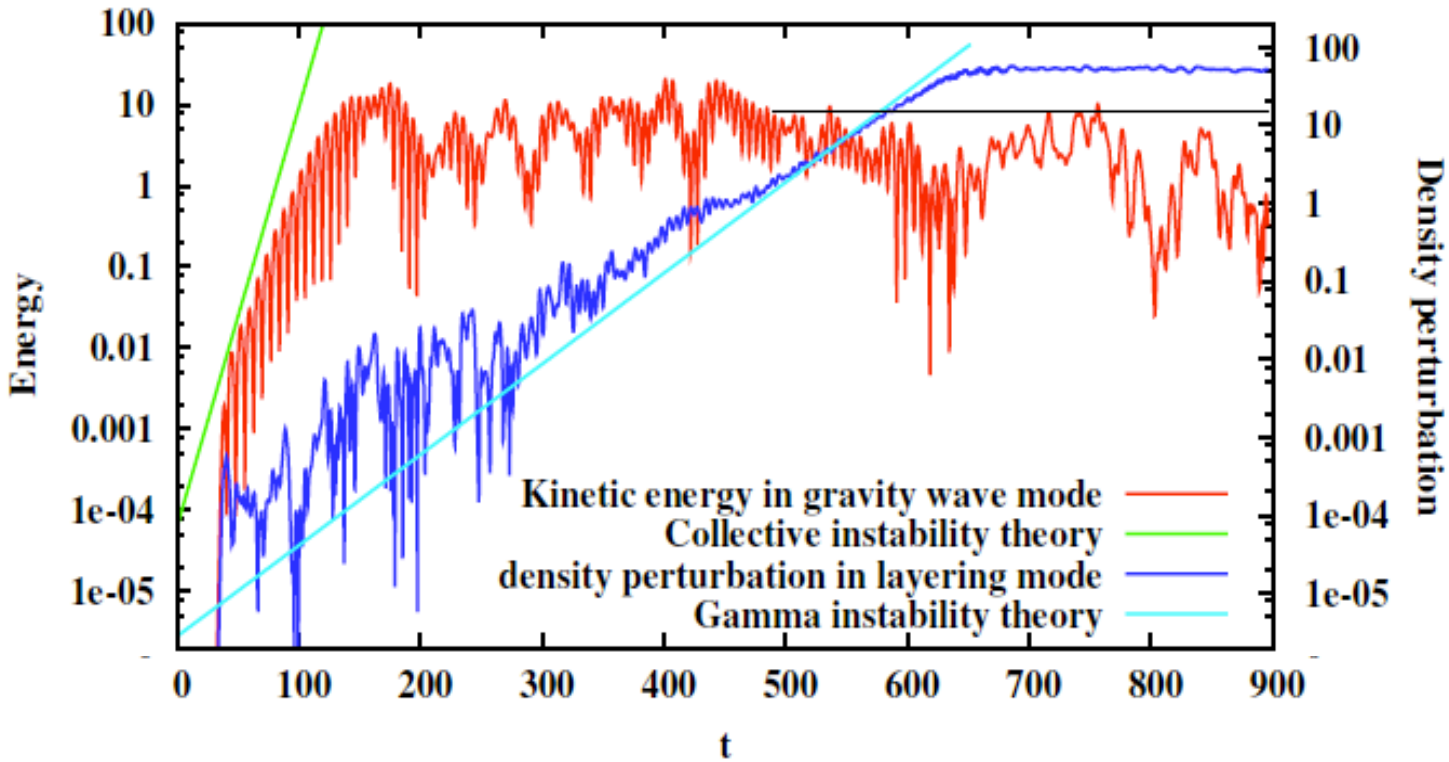
- Stability diagram: the Traxler "flower plot" shows the real part of solutions of the mean-field cubic $\Lambda^3 + a\Lambda^2 + b\Lambda + c = 0$ as functions of wavenumber



Mean-field theory: proof of concept

Comparison of the growth rates with numerical simulation shown earlier:

$$\mathrm{Pr} = 7,\ \tau = 1/3,\ R_0 = 1.1$$



Mean field theory works!

Stellmach et al. 2011

Application to stars

- To predict whether large-scale instabilities develop in stars , we can use the Brown et al. (2013) model to compute

$$\mathrm{Nu}_T = \mathrm{Nu}_T(R;\mathrm{Pr},\tau)$$

$$\gamma = \gamma(R;\mathrm{Pr},\tau)$$

- This implies $\mathrm{Nu}_T = 1 - \left\langle \hat{w}\hat{T} \right\rangle = 1 + \dfrac{C_B^2 \lambda^2}{k_h^2(\lambda + k_h^2)}$

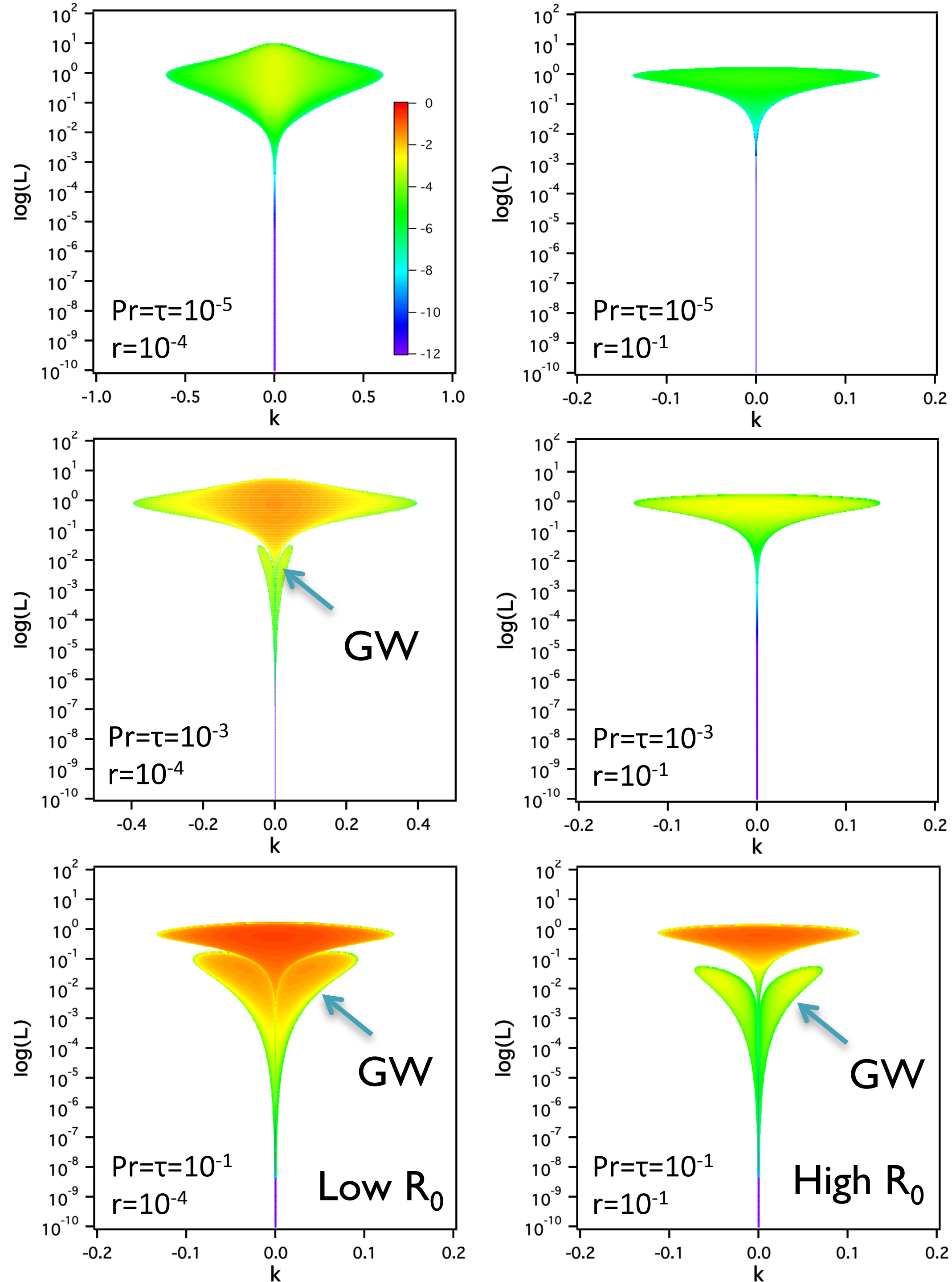
$$\gamma = \frac{-1 + \left\langle \hat{w}\hat{T} \right\rangle}{-\tau R_0^{-1} + \left\langle \hat{w}\hat{C} \right\rangle} = \frac{1 + \dfrac{C_B^2 \lambda^2}{k_h^2(\lambda + k_h^2)}}{\tau R_0^{-1} + \dfrac{C_B^2 \lambda^2}{R_0 k_h^2(\lambda + \tau k_h^2)}}$$

Predictions for fingering convection, astrophysical regime:

- **No layering instability**!
- **Gravity waves:**
  - For low R_0, gravity waves exist for Pr down to 10^{-3}, but not lower.
  - For higher R_0, or low Pr, τ gravity waves are absent

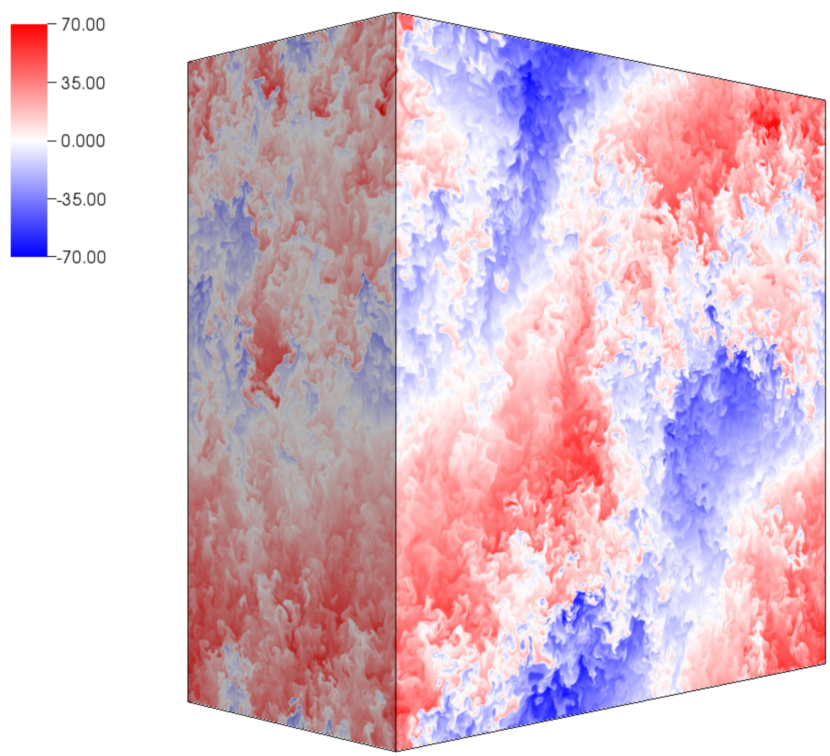
Large-scale structures do not emerge in stars (where Pr < 10^{-6})

From Garaud et al. 2015



Gravity waves ?

- We indeed find gravity wave excitation in the predicted region of parameter space (e.g. Pr ~ τ ~ 0.01)



However, it is not clear whether fingering convection with this low R_0 can ever be triggered deep inside degenerate regions of stars (Garaud et al. 2015)

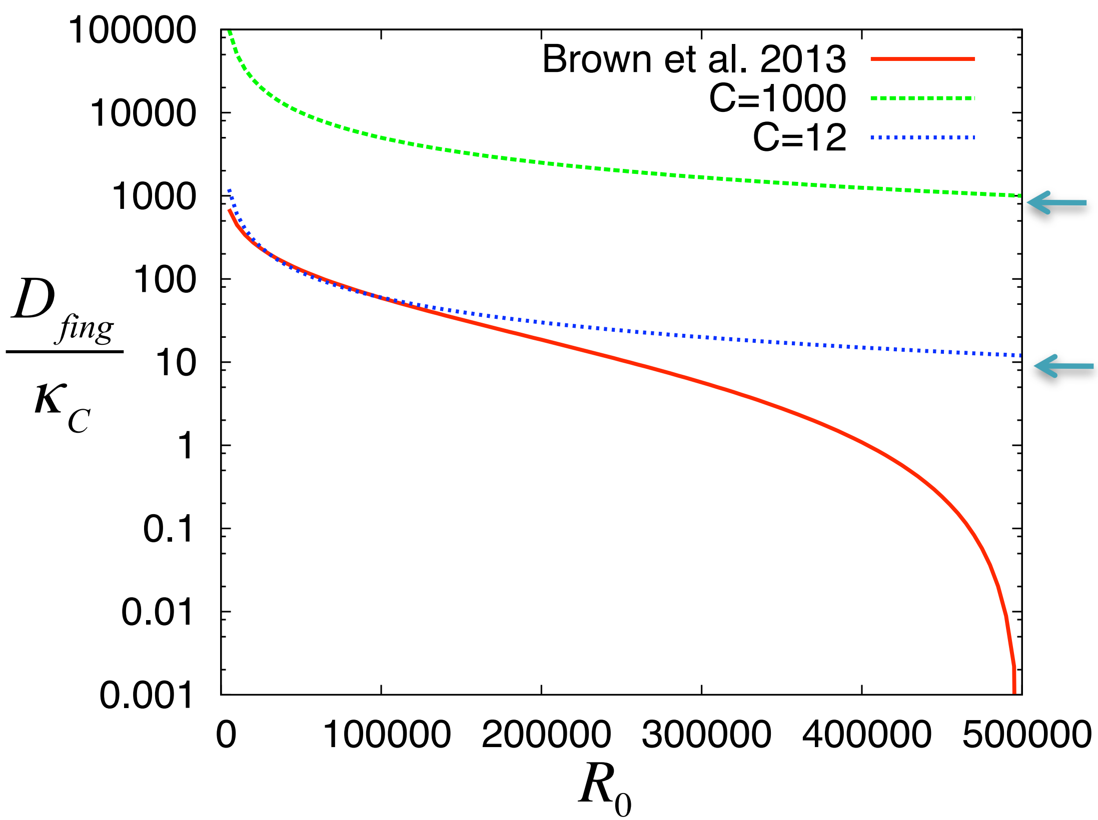
$$\mathrm{Pr} = \tau = 0.03, R_0 = 1.11$$

Implications:

The Brown et al. 2013 model applies!

Implications for RGB stars.

Mixing by fingering probably cannot explain RGB abundances (cf. Denissenkov 2010).



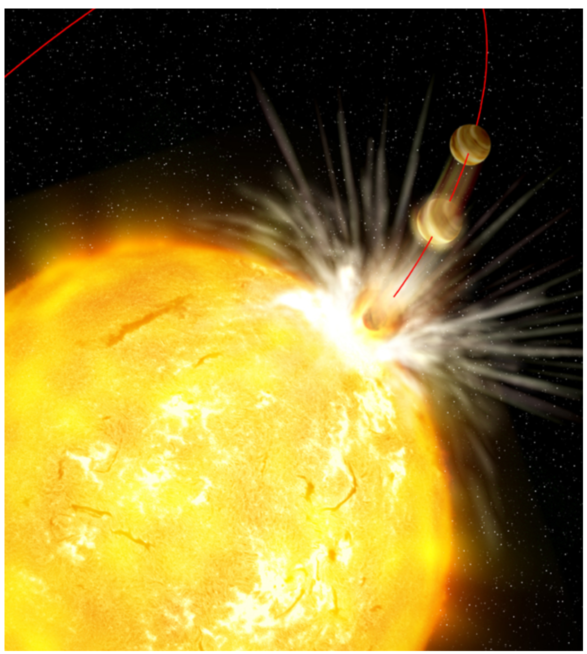
Required by observations (Charbonnel & Zahn 2007)

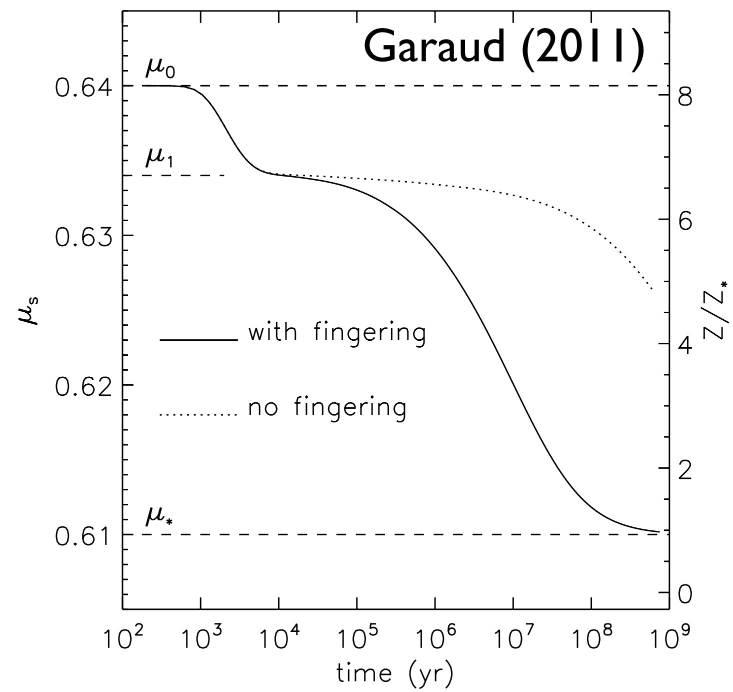
Original Kippenhahn et al. 1980 proposal.

$$\frac{D_{fing}}{\kappa_C} \propto \frac{C}{\tau R_0}$$

Implications for planetary accretion onto MS stars

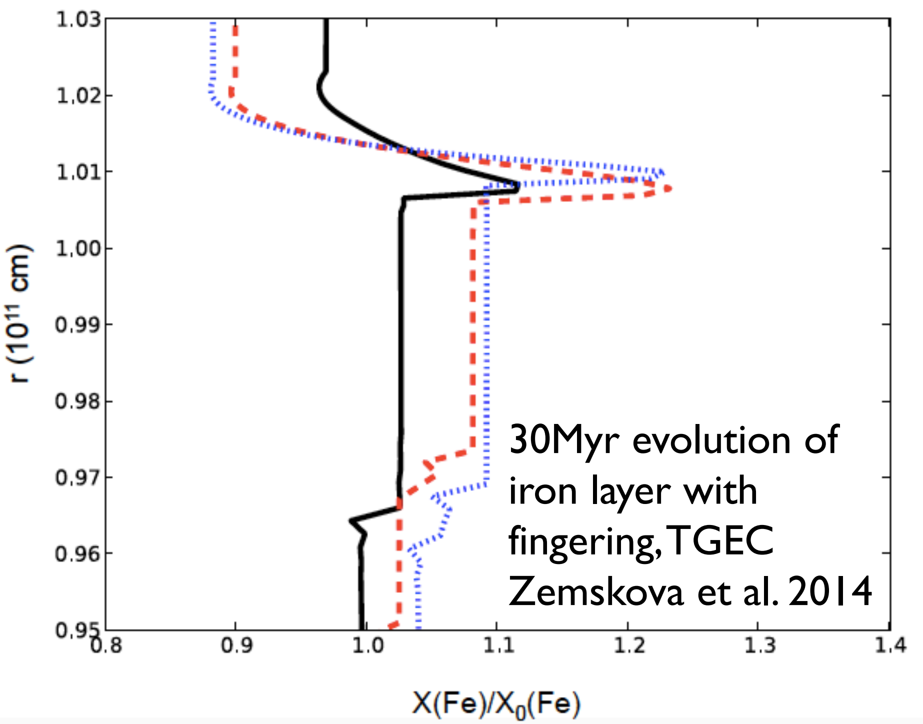
Fingering convection explains why MS stars that have accreted planets do not show evidence for higher metallicity (cf. Vauclair 2004).

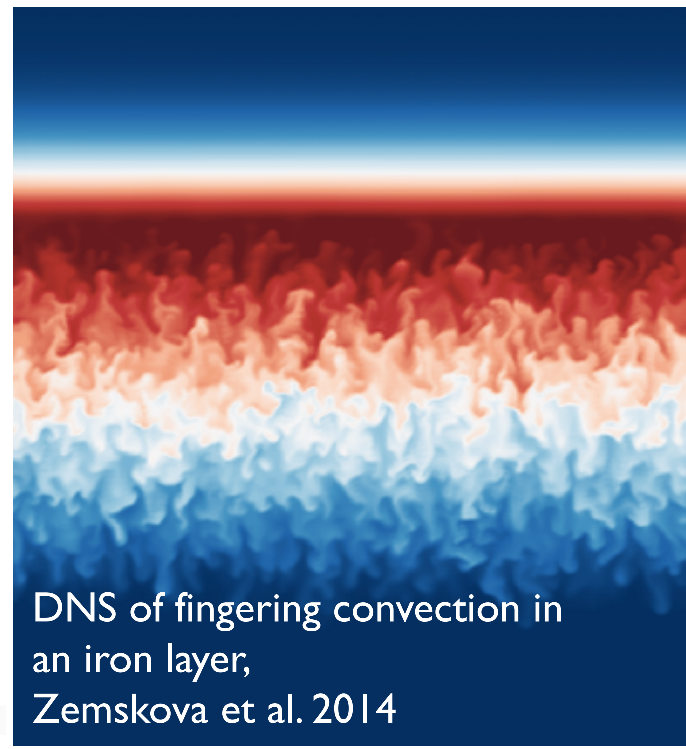




Implications for element layers

Fingering convection strongly moderates the formation of element-rich layers in intermediate mass stars (Zemskova et al. 2014); convective layers probably do not form (TBC)





Take-home messages

Basic fingering instability:

- Fingering instabilities can occur in a wide variety of situations in stars, whenever unstable mu-gradient develop
- Fingers are typically small scale (~10-100m)
- Saturation occurs because of secondary shearing instabilities in between up- and down fingering.
- Nonlinear fluxes can be predicted semi-analytically using "linear" theory of Brown et al. (2013), or analytically using their asymptotic model.

Take-home messages

Mean-field instabilities:

Under some circumstances, larger-scale structures (e.g. staircases, large-scale gravity waves) form in fingering convection. This can be studied using mean-field theory (Radko 2003; Traxler et al. 2011).

We find that at astrophysical parameters

- *No layering instability*
- Gravity waves only excited at intermediate Pr (degenerate matter), very low density ratio
- For non-degenerate stellar interiors, neither layers nor gravity waves are excited.