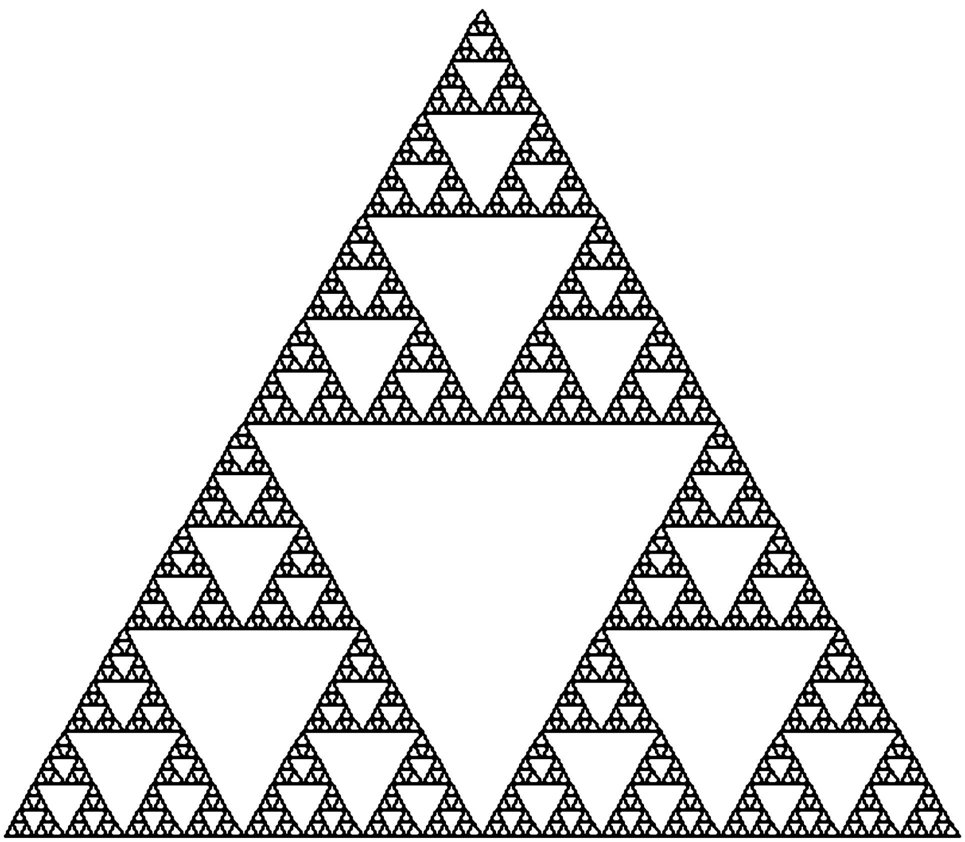
Self-Similar Sierpinski Fractals



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Science Fair 2008

Self-Similar Sierpinski Fractals

Introduction

For my science fair project I am studying fractals. Several of the fractals I am studying were discovered by Wacław Sierpiński. First, I discuss an important property of these objects, self-similarity, as well as its counterpart, recursion. Then I explain my methods for creating fractals.

Next I show my derivations of scaling factor, another facet of self-similar objects. I derive and present a summation formula that works for all regular polygons and various closed-form formulas that work for circles, polygons with even numbers of sides, polygons with any number of sides, and stars.

Following is a brief digression on the area of fractals, focusing on the Sierpinski triangle.

Subsequently, I introduce my primary topic, fractal dimension. I give an explanation of the definition of fractal dimension, yielding a formula for computing it. I hypothesized that fractal dimension would increase as the number of sides increases. I created and present a table of fractal dimension that shows that my hypothesis is wrong.

I then consider the limit of fractal dimension as the number of sides goes to infinity. I created a new hypothesis that the limit goes to one. I used various limit theorems, such as L'Hopital's rule, to prove my hypothesis correct.

Self-Similarity

A self-similar object is an object that is composed of multiple smaller copies that look exactly the same as it.

The **scaling factor,** s, is ratio of the size of the object to the size of the copies. The largest copy of a self-similar object is s times bigger than the next largest.

The **number of copies,** n, is also an important parameter for self-similar objects. It is how many smaller copies of the object comprise the object.

Recursion is a property of some procedures, meaning that they call themselves. It is much like self-similarity; in fact, one way of creating self-similar fractals is through recursive code.

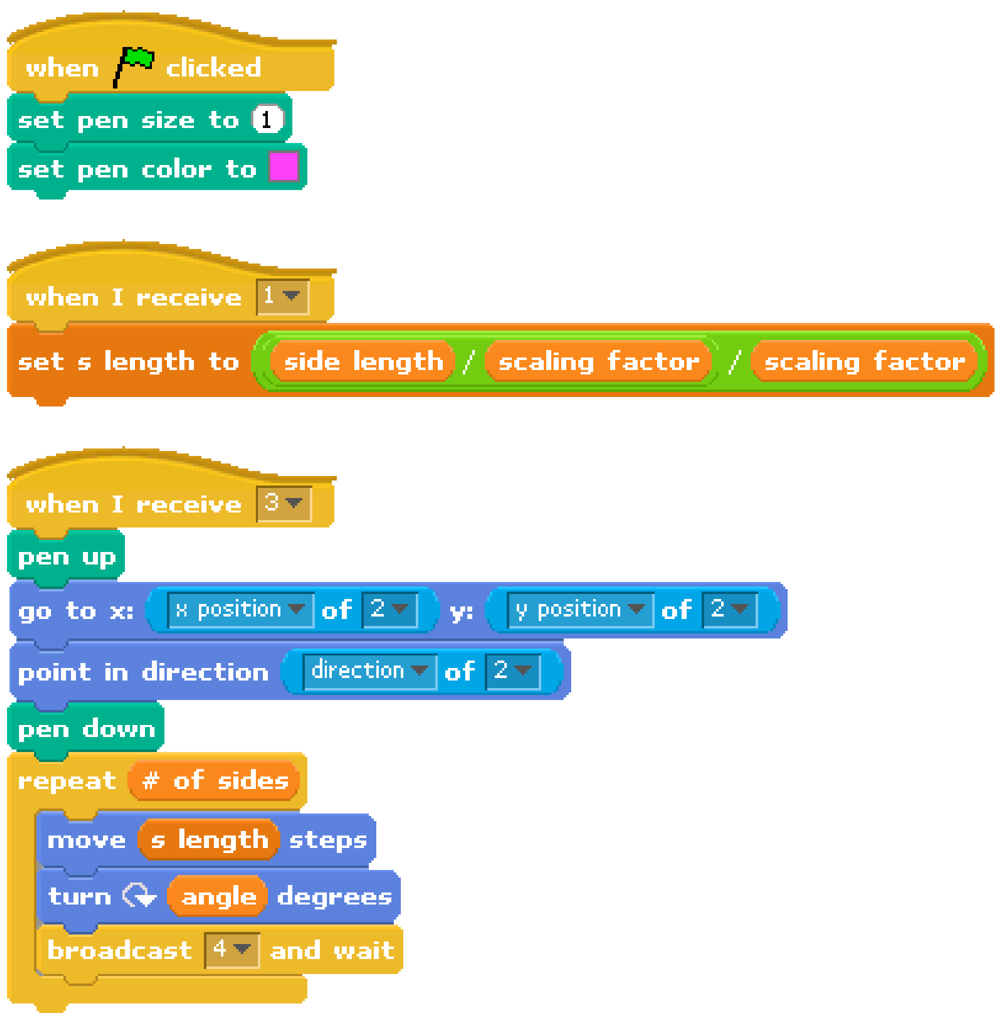
Programs to make Sierpinski objects

I wrote programs to generate pictures of fractal using Scratch, a free graphical programming language available from MIT.

Recursive program

Scratch doesn't support recursion, in that the same script cannot be running twice at the same time. So, in order to get around the problem, I had to create several copies of the script, one for each level of recursion.

This picture shows the scripts for level 3 of the recursion.

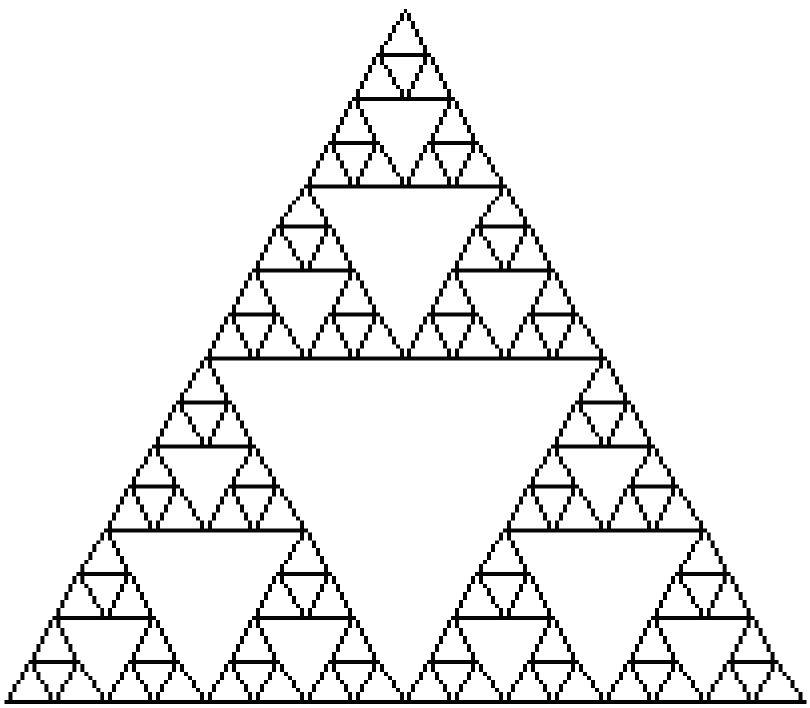


This is the initialization code that begins the recursive program

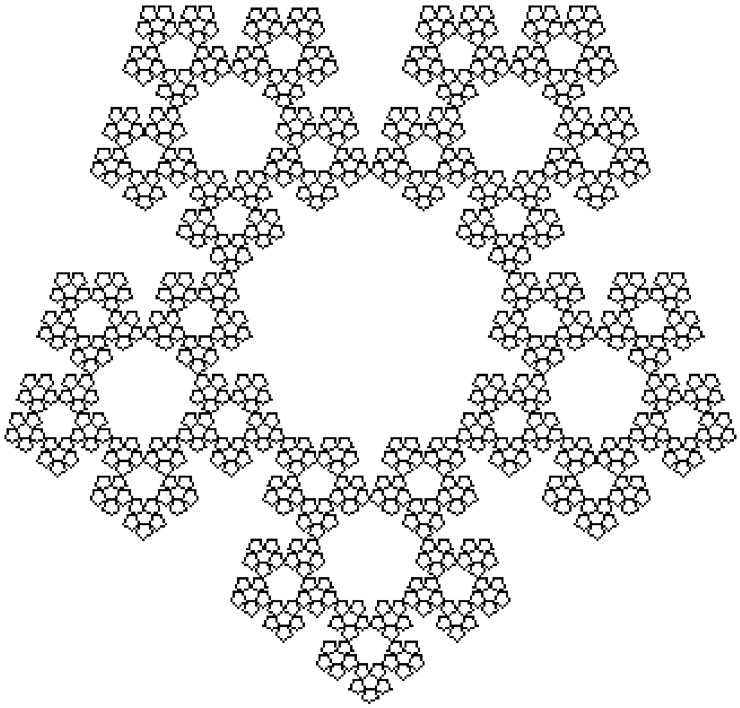
This code computes and (via a printing program of Kevin's) outputs fractal dimension

Fractal Images

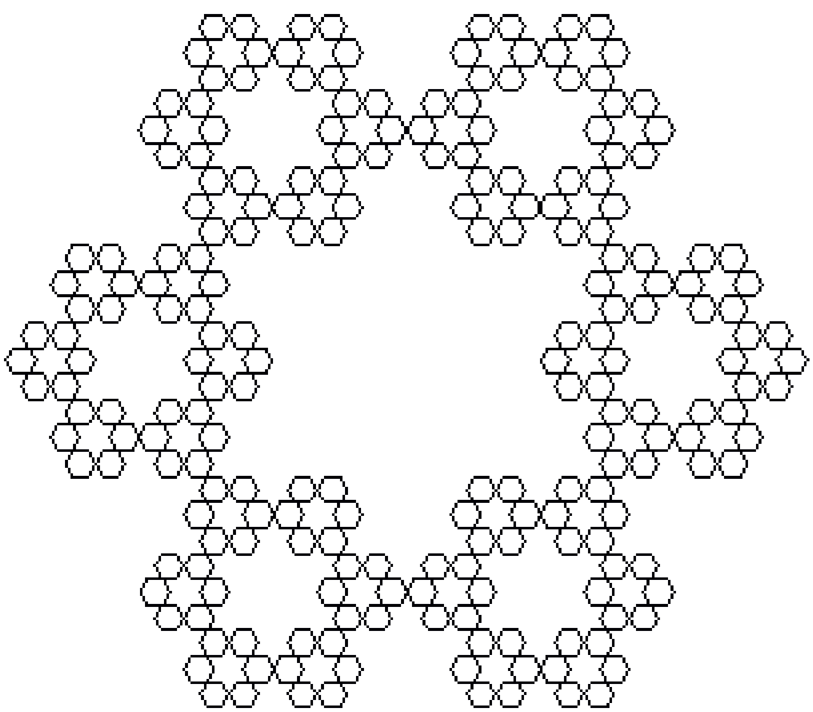
This section has many fractal images created using the recursive program shown previously.



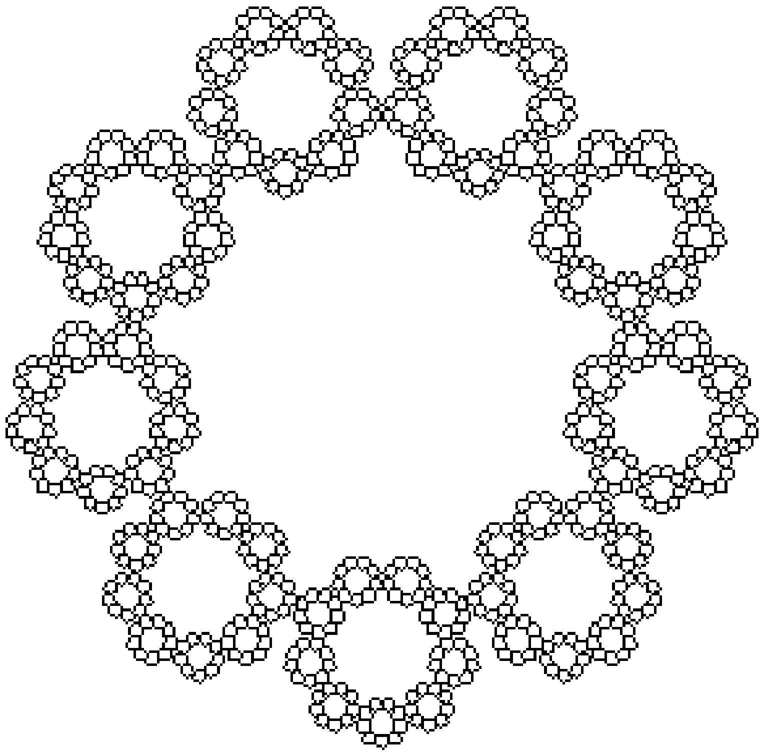
Sierpinski Triangle



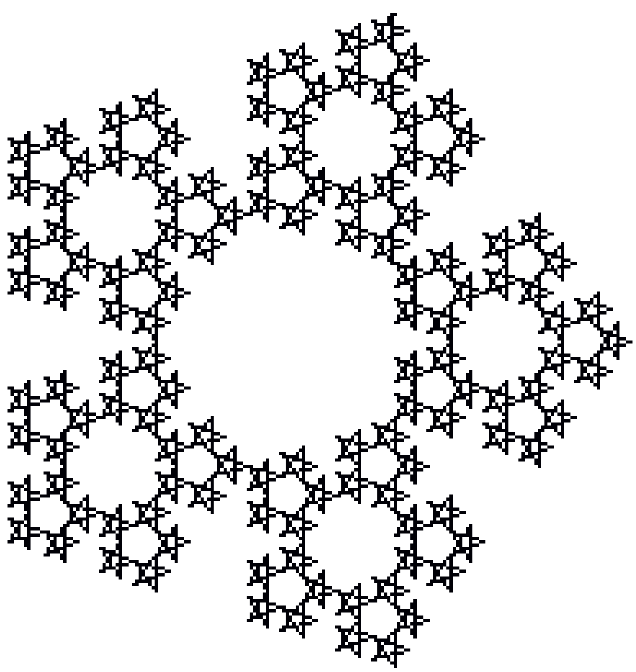
Sierpinski Pentagon



Sierpinski Hexagon



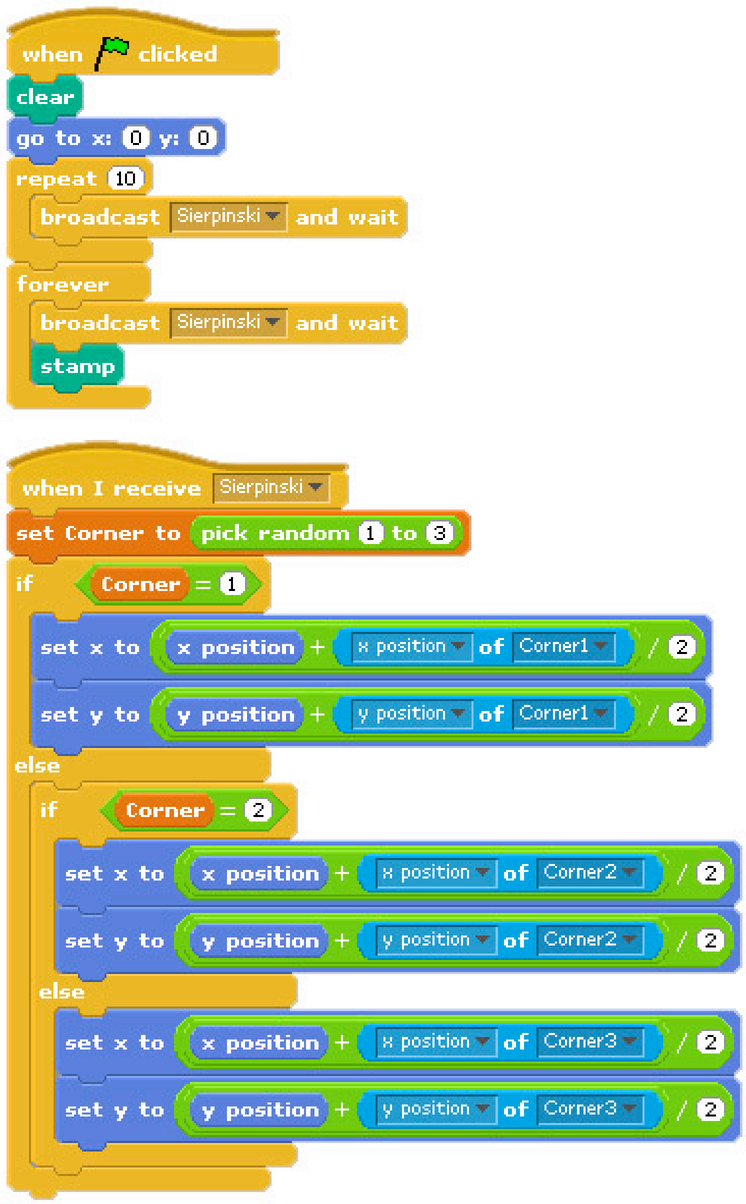
Sierpinksi Nonagon



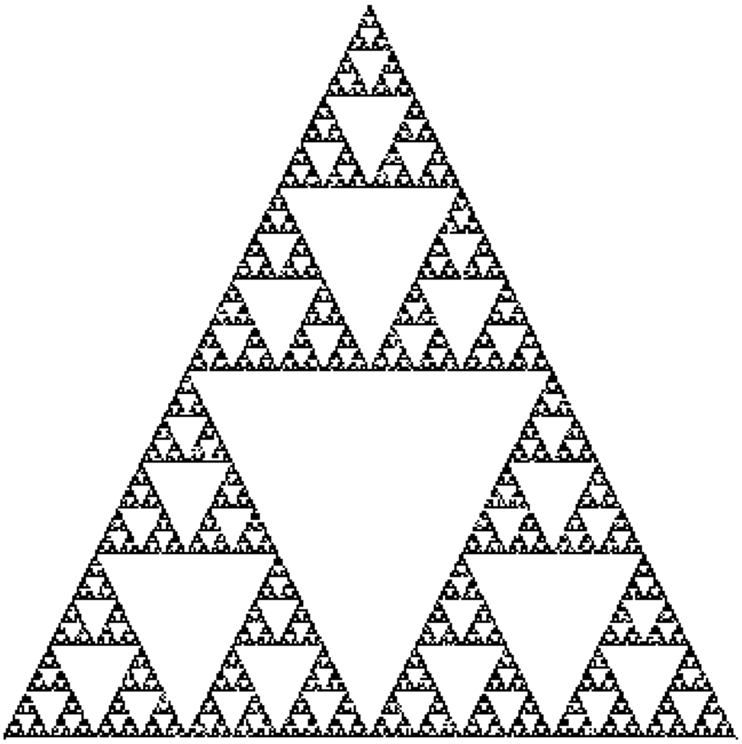
Fractal 5-pointed Star

Chaos Game

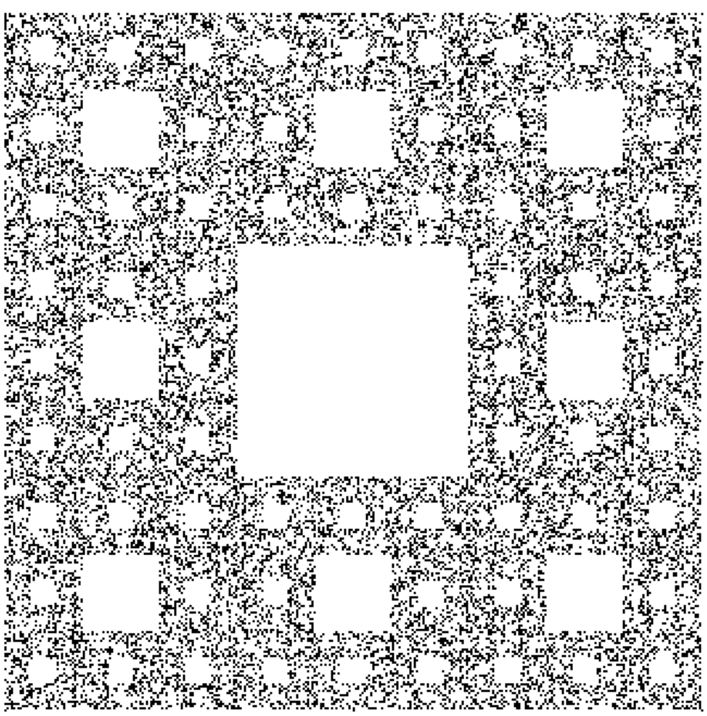
A non-recursive way of getting Sierpinski polygons is by using what is known as the **Chaos Game**. This involves randomly choosing one of the vertices of the object (hence the name “chaos”), averaging the current position with the position of that vertex, leaving a dot at the new position, and then repeating. For a Sierpinski triangle, a simple average is fine, but for higher shapes, you need a weighted average. The current position gets a weight of 1/s (s is the scaling factor), and the vertex gets a weight of 1 – (1/s).



Chaos game code



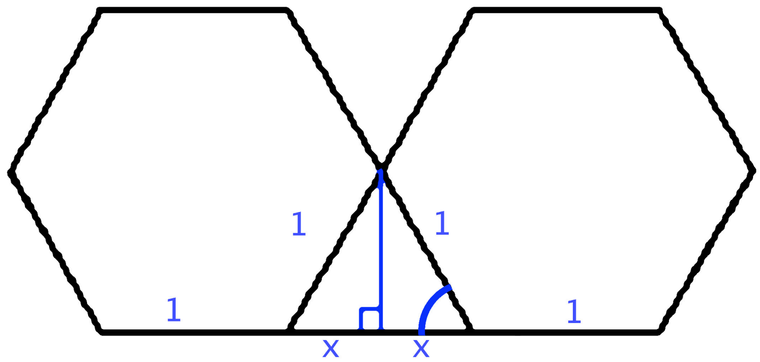
Sierpinski Triangle — Chaos Game



Sierpinski Carpet — Chaos Game, a distorted octagon

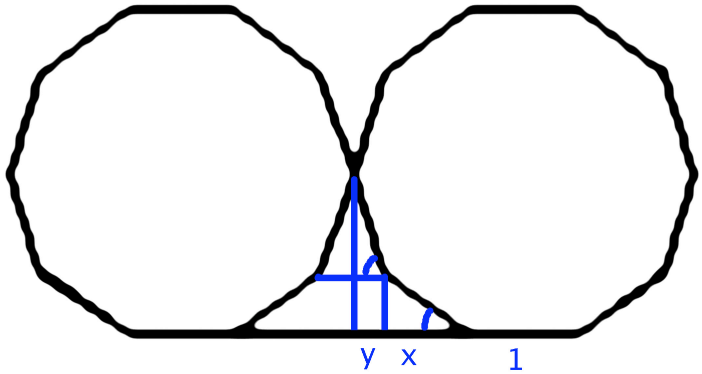
Summation Formula for Scaling Factor

I want to compute a scaling factor such that the smaller polygons inside the big one just touch each other without overlapping. For triangles and squares, the scaling factor is just 2.



In this diagram the small sides are length 1 and the big side is 2x+2, which would be the scaling factor. So, what is x? It is one side of a right triangle, whose hypotenuse is 1. The angle adjacent to x is the exterior angle $360^o/n$, where n is the number of sides. So $x=\cos(360^o/n)$. This formula works for squares through octagons.

If we look at decagons, the scaling factor is 2y+2x+2. As before $x=\cos(360^o/n)$, and $y=\cos(2*360^o/n)$, because the second angle has turned 2 exterior angles from the base.



The general formula for the scaling factor of an n-gon is

$$2 \sum_{k=0} \cos(360^o\, k/n) \,,$$

where the sum continues only as long as the cosine term is positive.

I created a Scratch program to compute scaling factor. Here are the results for certain numbers:

# of vertices	Scaling Factor
3	2.00000
4	2.00000
5	2.61803
6	3.00000
7	3.24698
8	3.41421
9	3.87939
10	4.23607
11	4.51334
12	4.73205
13	5.14811
100	32.82052
1000	319.30884
10000	3184.0988

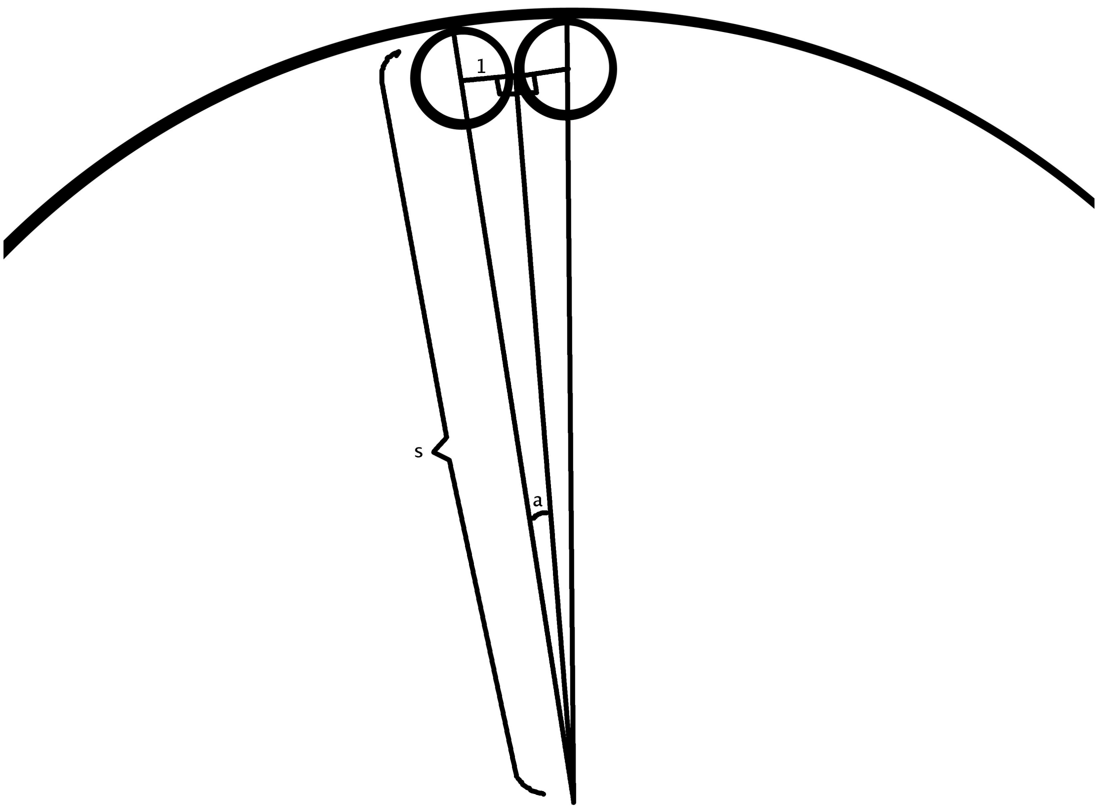
Closed-Form Formula for Scaling Factor

Circles

For large polygons, the summation scaling factor formula can take ages to compute. I have investigated fractal circles as an approximation to large polygons, in an attempt to determine if there is a limit (such as 1) to fractal dimension as the number of sides goes to infinity. I have derived the formula for scaling factor of a circle (that has n smaller circles placed around it). It is

$$s=1+(1/\sin(180^{o}/n)).$$

Heres how I got it:



Each small circle takes up 360o/n of the large circle, so a=180o/n. Trigonometrically speaking, the opposite of a is 1, and its hypotenuse is s–1, so

$$\sin(a)=1/(s-1).$$

Solving for s gives us

$$s=1+(1/\sin(180o/n)) .$$

I made a table comparing the summation formula to the circle formula:

# of vertices	Scaling Factor	Circle Scaling Factor
3	2.00000	2.15470
4	2.00000	2.41421
5	2.61803	2.70130
6	3.00000	3.00000
7	3.24698	3.30476
8	3.41421	3.61313
9	3.87939	3.92380
10	4.23607	4.23607
11	4.51334	4.54947
12	4.73205	4.86370
13	5.14811	5.17858
100	32.82052	32.83623
1000	319.30884	319.31041
10000	3184.0988	3184.09891

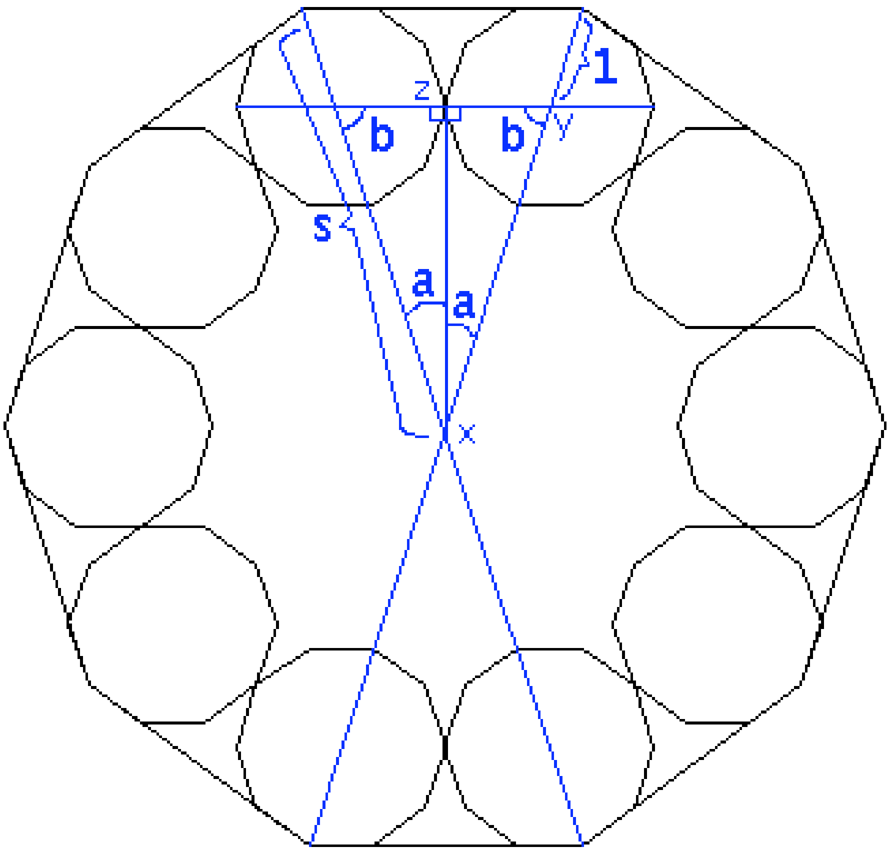
I noticed two things about this table. One is the fact that the error reduces as n increases. Another is that at n=6 and n=10, there is a perfect match. I thought that this match might occur every increment of 4. I tried it with 14 and 18:

14	5.49396	5.49396
18	6.75877	6.75877

It works for 14 and 18. Why is this?

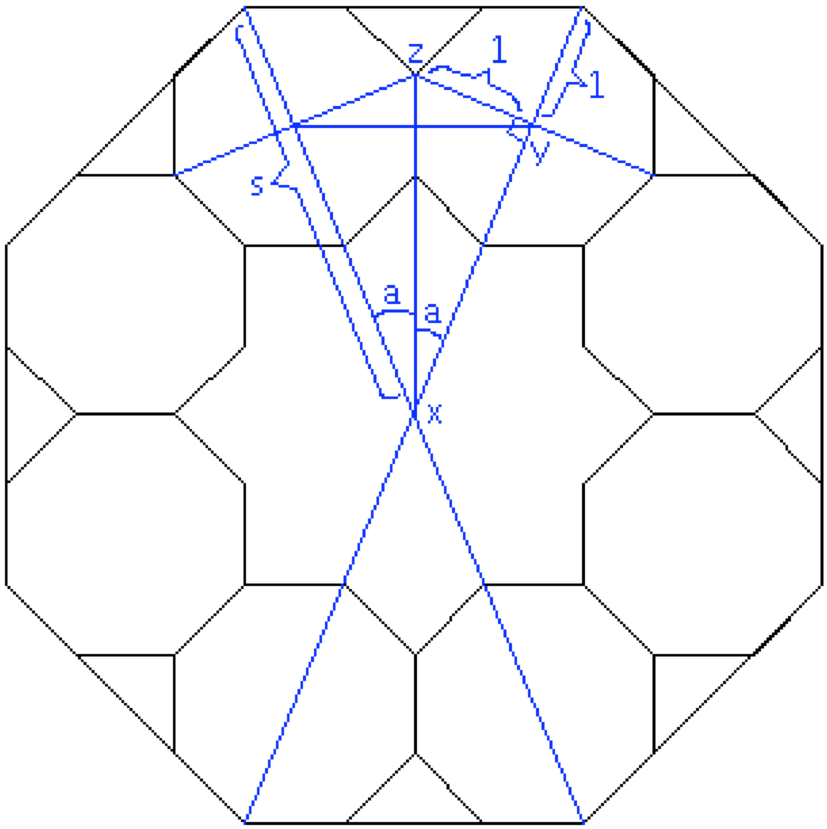
Even Polygons

What if I tried using the circle technique on a decagon? Here's what happened:



In this decagon, the line through the two centers goes through point z, so the same derivation as for circles applies.

But what if the fractal polygon has a multiple of four sides? Here's what happened for an octagon:



In this octagon, a=180°/n. The tangent of a is 1/(s–1) because tangent is opposite (yz) over adjacent (xy):

$$\tan(a)=1/(s-1)$$

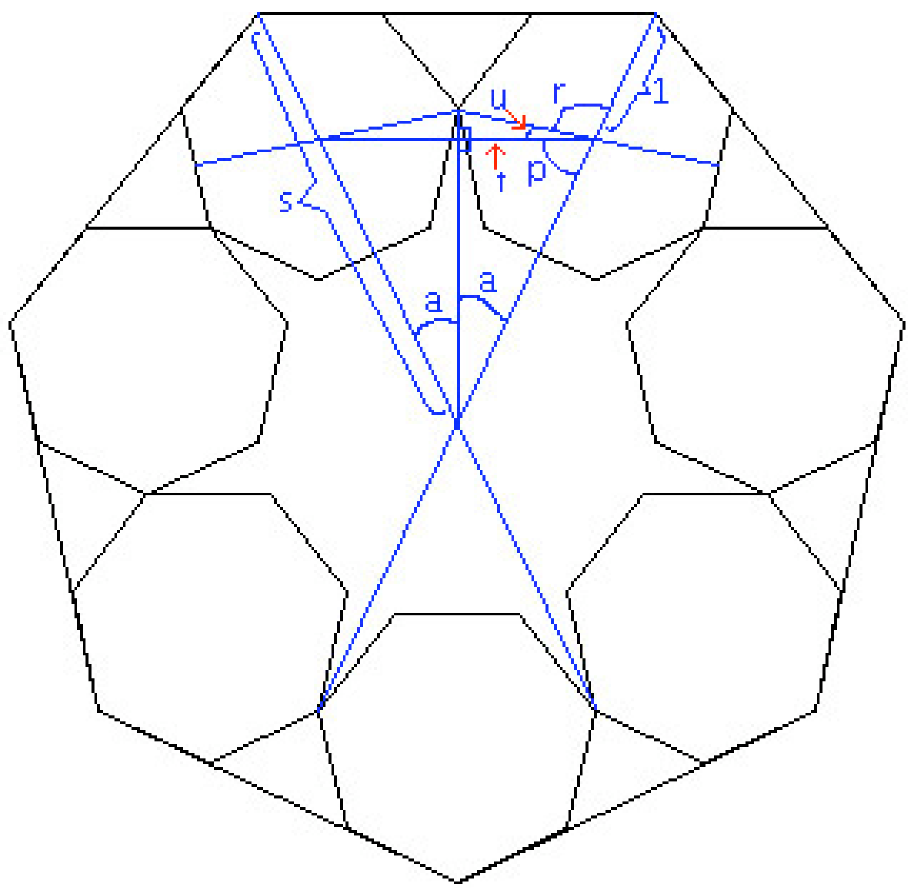
Solving for s gives us

$$s=1 + 1/\tan(180°/n) .$$

This works for squares, octagons, dodecagons, etc.

Odd Polygons

But what about fractal polygons with odd numbers of sides? Here's what happened for a heptagon:



In this heptagon, a=180º/n because it occupies 1/2 of one of n equal sections of the circle surounding the main polygon. Also, sin(a)=t/(s–1) because sine is opposite over hypotenuse, so

$$s=1 + t/\sin(180^\circ/n) .$$

Next, t=cos(u), since cosine is adjacent over hypotenuse. Now u=180º – (r+p), as these three angles share a straight line. As for p, it is simply 90º – a, as triangle angles add to 180º. But r is more complicated—it is a multiple of 360º/n, which is the number of degrees taken by one side, as it takes up a number of sides. This multiple, v, is one plus the number of sides before an intersection (the one is the top side) (sides before intersection is: 0 for 3–4-gons, 1 for 5–8-gons, 2 for 9–12-gons, etc.), which increments every four, making its formula

$$v=\text{floor}((n-1)/4) + 1$$

(the floor of something is it rounded down). Plugging this in (multiplying by $360^\circ/n$) gives

$$r=(360^\circ*(\mathrm{floor}((n-1)/4)+1))/n\ .$$

More substitution gives

$$t=\cos(180^\circ-((90^\circ-180^\circ/n)+((360^\circ*(\mathrm{floor}((n-1)/4)+1))/n))$$

Next, since $\cos(180^\circ-\text{something})=-\cos(\text{something})$,

$$t=-\cos((90^\circ-180^\circ/n)+ ((360^\circ*(\mathrm{floor}((n-1)/4)+1))/n))$$

This is now of the form $t=-\cos((90^\circ-k)+f)$, which rearranges to $t=-\cos(90^\circ+(f-k))$. But $-\cos(90^\circ+\text{something})=\sin(\text{something})$, so

$$t=\sin(((360^\circ*(\mathrm{floor}((n-1)/4)+1))/n) - 180^\circ/n)$$

Both of these terms are over n, so n can be factored out,

$$t=\sin\left(\frac{(360^\circ*(\mathrm{floor}((n-1)/4)+1) - 180^\circ}{n}\right)$$

If we multiply out $360^\circ*(\text{something}+1)$, we get

$$t=\sin\left(\frac{(360^\circ*\mathrm{floor}((n-1)/4)+360^\circ) - 180^\circ}{n}\right)$$

Rearranging $(360^\circ*\text{something}+360^\circ)-180^\circ$ gives

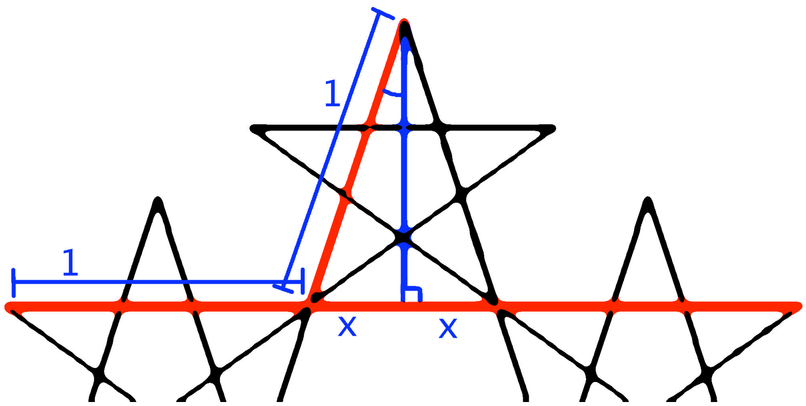
$$t=\sin\left(\frac{180^\circ+360^\circ*\mathrm{floor}((n-1)/4)}{n}\right)$$

If this is substituted into $s=1+(t/\sin(180^\circ/n))$, we get

$$s=1+\frac{\sin\left(\frac{180^\circ+360^\circ*\mathrm{floor}((n-1)/4)}{n}\right)}{\sin(180^\circ/n)}$$

This imposing formula works for all (not just odd) fractal polygons.

Stars



In the diagram above, the scaling factor of a star is 2x+2, where x is the sine of the angle shown. That angle is half the interior angle, which is 180º minus the exterior angle, which in turn is 720º/n. So the formula is

$$s = 2 + 2(\sin(90^\circ - 360^\circ/n))$$

which can be further simplified to

$$s = 2 + 2(\cos(360^\circ/n))$$

This formula works for 5, 7, and 9 sides, but we start getting overlap at 11 sides.

What is the area of a Sierpinski Triangle?

To find the area of a Sierpinski Triangle, I'll consider an alternative method of construction, in which one starts with a filled-in triangle and recursively removes the central triangle.

When we punch out the middle triangle, the area is ¾ of the original. When we do this again to the subtriangles, the area is $(¾)^2$. So the area after T levels is $(¾)^T$. The limit of this as T goes to infinity is the area of the Sierpinski Triangle. This limit is **zero** because

- It cannot be negative as multiplying positive numbers gets you a positive number.
- (¾)T always decreases as T increases.
- It cannot be positive. If it were, it would be some number c>0. This being a limit means that we can get arbitrarily close to c, if T becomes large enough. We can get to less than 4c/3, but ¾ of this is less than c, so the limit must be less than c, meaning that c cannot be the limit.

What is Fractal Dimension?

Concept

To explain fractal dimension, let's consider a line. To double it in length we need two copies; for a square we need four copies, and a cube requires eight. We can see that n, the number of copies, is s, the scaling factor, to the power of D, the dimension: $n=s^D$. Solving this equation for D would give us a definition of dimension for a self-similar object.

Derivation

Starting with $n=s^D$, and taking the logarithm of both sides, we get

$$\log(n) = \log(s^D) .$$

Using the property of logarithms, $\log(a^b)=b \log(a)$, we get

$$\log(n) = D \log(s) .$$

Dividing both sides by log(s) gives us

$$D = \log(n) / \log(s) ,$$

which will be our definition of fractal dimension.

How does the Fractal Dimension of Sierpinski Objects Vary with the Number of Sides?

Hypothesis: The Dimension will Increase as the Number of Sides Increases.

I created a Scratch program that tells me the fractal dimension and scaling factor of an object. The table below shows the results.

Table of Fractal Dimensions

# of vertices	Scaling Factor	Fractal Dimension
3	2.00000	1.58496
4	2.00000	2.00000
5	2.61803	1.67228
6	3.00000	1.63093
7	3.24698	1.65226
8	3.41421	1.69343
9	3.87939	1.62076
10	4.23607	1.59499
11	4.51334	1.59113
12	4.73205	1.59867
13	5.14811	1.56530
100	32.82052	1.31913
1000	319.30884	1.19798
10000	3184.0988	1.14188

This table shows that my hypothesis was wrong—in fact, the dimension decreases.

Limits

What is the limit of fractal dimension as n goes to infinity? I used Maple, a symbolic algebra program, in an attempt to determine this. My hypothesis, based on the table, is the limit is 1. Maple could not determine the limit, as floor is not differentiable. When I replaced floor((n–1)/4) with n/4 – c, a similar function (for a given c, the functions are equal every four values). With this new function, Maple gave a limit of 1. Now, of course, I must prove this.

First, can I make the formula easier to take the limit of? The top half of the scaling factor looks messiest, so I'll concentrate on that.

$$t = \sin\left(\frac{180^\circ + 360^\circ * \text{floor}((n-1)/4)}{n}\right)$$

This should be changed to the n/4–c version, to allow easier manipulation:

$$t = \sin\left(\frac{180^\circ + 360^\circ * (n/4 - c)}{n}\right)$$

Multiplying out the top gives

$$t = \sin\left(\frac{180^\circ + 90^\circ * n - 360^\circ * c}{n}\right)$$

I can split this fraction to get

$$t = \sin\left(90^\circ + \frac{180^\circ - 360^\circ * c}{n}\right)$$

Next, since $\sin(90^\circ + x) = \cos(x)$, this is

$$t = \cos\left(\frac{180^\circ - 360^\circ * c}{n}\right)$$

I can factor out 180° to get

$$t = \cos\left(180^\circ * \frac{1-2*c}{n}\right)$$

Now let's look at the scaling factor as a whole. It is s=1 + t/sin(180º/n). Let's define v = sin(180º/n), so that s= 1+t/v. Putting the 1 into the fraction gives

$$s = \left(\frac{t+v}{v}\right)$$

As fractal dimension f=log(n)/log(s), we have

$$f = \left(\frac{\log(n)}{\log\left(\frac{t+v}{v}\right)}\right)$$

We can simplify the bottom of this to get

$$f = \left(\frac{\log(n)}{\log(t+v) - \log(v)}\right)$$

What happens to each of these terms as n goes to ∞? Well, v=sin(180º/n), which goes to sin(0º), which is 0, so log(v) goes to –∞. On the other hand, t goes to cos(0º), which is 1, so log(t+v) goes to log(1), which is 0. So we can take the limit of something simpler:

$$\lim f = \lim \left(\frac{\log(n)}{-\log(v)}\right)$$

This is ∞/∞, so we need to apply L'Hopital's rule: if lim f(x)/g(x) is undefined for substitution, by being ∞/∞ or 0/0, then

$$\lim f(x)/g(x) = \lim f'(x)/g'(x) .$$

Since this involves derivatives, we need to convert from degrees to radians. The limit we want to take is

$$\lim f = \lim \left(\frac{\log(n)}{-\log(\sin(\pi/n))} \right)$$

Let’s define x to be π/n, so x goes to 0 as n goes to infinity, and we need to take the limit of

$$\lim f = \lim \left(\frac{\log(\pi/x)}{-\log(\sin(x))} \right)$$

We can simplify the top

$$\lim f = \lim \left(\frac{\log(\pi) - \log(x)}{-\log(\sin(x))} \right)$$

Then we use L’Hopital’s rule, giving

$$\lim f = \lim \left(\frac{-1/x}{-\cos(x)/\sin(x)} \right)$$

which simplifies to

$$\lim f = \lim \left(\frac{\sin(x)}{x \cos(x)} \right)$$

As x goes to 0, cos(x) goes to 1, so we can simplify further to

$$\lim f = \lim \left(\frac{\sin(x)}{x} \right)$$

We can use L'Hopital's rule again to get

$$\lim f = \lim \left(\frac{\cos(x)}{1} \right)$$

but cos(x) goes to 1 as x goes to 0, so we have

$$\lim f = 1$$

Conclusion

The fractal dimension has a downward trend, though it is not always decreasing. Its limit as the number of sides goes to infinity is 1.

Further Research

The scaling factor formula for stars is incomplete. A formula for the scaling factor for all stars still needs to be found.

Acknowledgments

I got the Scratch programming language from http://scratch.mit.edu

Kevin Karplus, my Dad, helped me understand the math involved.

References

Chaos and Fractals: New Frontiers of Science by Peitgen, Jürgens, Saupe

http://math.bu.edu/DYSYS/chaos-game/chaos-game.html