AMS 147 Computational Methods and Applications **Course Project**

Use RK4 to solve the initial value problem of van der Pol equation

$$\begin{cases} y'' + \mu\left(y^2 - 1\right)y' + y = 0 \\ y(0) = 0\,, \quad y'(0) = 4 \end{cases}$$

We write the second order ODE as a first order ODE system

$$\begin{cases} \dfrac{d}{dt}\vec{W}(t) = \vec{F}\left(\vec{W}(t), t\right) \\ \vec{W}(0) = \vec{W}_0 \end{cases}$$

where $\vec{W}(t) = \left(y(t),\ y'(t)\right)$. The initial value is $\vec{W}(0) = \left[0,\ 4\right]$.

Task 1: $\boxed{\text{Numerical error estimation and selection of a time step}}$

For μ = 3, solve for $\vec{W}(t)$ from t = 0 to t = 100.

Run simulations using time steps

$$h = \frac{1}{2^3},\ \frac{1}{2^4},\ \ldots,\ \frac{1}{2^9}$$

Do numerical error estimations at t = 100 (see Appendix for numerical error estimation)

Plot the 2-norm of estimated error vs. time step h. Use logarithmic scales for both the horizontal axis (time step) and the vertical axis (2-norm of the estimated error).

Select a time step h_c such that

2-norm of estimated error $< 0.5 \times 10^{-4}$

Use the selected time step h_c in all of the simulations below.

Task 2: $\boxed{\text{Study the period of limit cycle}}$

As $t \to +\infty$, the solution $\vec{W}(t) = \left(y(t),\ y'(t)\right)$ converges to a steady sate limit cycle.

<u>We assume</u> that $\vec{W}(t) = \left(y(t),\ y'(t)\right)$ has already converged to the limit cycle for $t \geq 70$.

The period of the limit cycle is the period of $y(t)$ for $t \geq 70$.

Find the period T of the limit cycle based on the numerical solution of $y(t)$ for $70 \leq t \leq 100$ (which is a set of discrete points).

Carry out the calculations for $\mu = [0.1 : 0.1 : 3]$.

Plot $\dfrac{T}{2\pi}$, the period of the limit cycle normalized by 2π, as a function of μ.

Task 3: Validate the assumption in **Task 2**

In **Task 2**, we find the period using the numerical solution of $y(t)$ for $70 \le t \le 100$.

Repeat the calculations and plotting in **Task 2** using numerical solution of $y(t)$ for

$120 \le t \le 150$

Do you see any difference in the plot of period vs. μ?

Do you think the assumption in **Task 2** is valid?

Task 4: Study the convergence of $\vec{W}(t) = \big(y(t),\ y'(t)\big)$ onto the limit cycle.

We assume that the trajectory of the limit cycle is described by the numerical solution of $\vec{W}(t) = \big(y(t),\ y'(t)\big)$ for $70 \le t \le 100$.

For $0 \le t \le 10$, find the distance from $\vec{W}(t) = \big(y(t),\ y'(t)\big)$ to the trajectory of the limit cycle.

***** Important *****:

To speed up the calculation, select only about 100 points from the numerical solutions for $0 \le t \le 10$, and calculate the distance from each point to the trajectory of the limit cycle. If you calculate the distance from every point in the numerical solution for $0 \le t \le 10$, the calculation will be too slow.

Carry out the calculations for $\mu = 0.1,\ 0.2,\ 0.3,\ 0.5,\ 1.0$

Plot the distance as a function of t for these values of μ.

Compare the convergence onto the limit cycle for these values of μ.

Appendix: Numerical error estimation in solving ODE systems

Consider solving the initial value problem

$$\begin{cases} \dfrac{d}{dt}\vec{W}(t) = \vec{F}\big(\vec{W}(t), t\big) \\ \vec{W}(0) = \vec{W}_0 \end{cases}$$

Let $\vec{W}(t)$ be the exact solution at t.

Let $\vec{W}_N\left(h\right)$ be the numerical solution at time = $N\,h$, obtained with time step h using RK4.

We have

$$\vec{W}_N\left(h\right)=\vec{W}\left(N\,h\right)+\vec{E}_N\left(h\right)$$
$$\vec{E}_N\left(h\right)=\vec{C}_4h^4+o\left(h^4\right)$$

To estimate the error, we run simulations with h and $\frac{h}{2}$.

$$\vec{W}_N\left(h\right)=\vec{W}\left(N\,h\right)+\vec{C}_4h^4+o\left(h^4\right)$$
$$\vec{W}_{2N}\left(\frac{h}{2}\right)=\vec{W}\left(N\,h\right)+\frac{1}{2^4}\vec{C}_4h^4+o\left(h^4\right)$$

==> $$\vec{W}_N\left(h\right)-\vec{W}_{2N}\left(\frac{h}{2}\right)=\left(1-\frac{1}{2^4}\right)\vec{C}_4h^4+o\left(h^4\right)$$

==> $$\vec{C}_4h^4=\frac{\vec{W}_N\left(h\right)-\vec{W}_{2N}\left(\frac{h}{2}\right)}{\left(1-\frac{1}{2^4}\right)}+o\left(h^4\right)$$

Since $\vec{E}_N\left(h\right)=\vec{C}_4h^4+o\left(h^4\right)$, we obtain

$$\vec{E}_N\left(h\right)\approx\frac{16}{15}\left(\vec{W}_N\left(h\right)-\vec{W}_{2N}\left(\frac{h}{2}\right)\right)$$

The norm of the estimated error is

$$\boxed{\left\|\vec{E}_N\left(h\right)\right\|\approx\frac{16}{15}\left\|\vec{W}_N\left(h\right)-\vec{W}_{2N}\left(\frac{h}{2}\right)\right\|}$$

where $\left\|\vec{W}_N\left(h\right)-\vec{W}_{2N}\left(\frac{h}{2}\right)\right\|$ denotes the 2–norm of $\vec{W}_N\left(h\right)-\vec{W}_{2N}\left(\frac{h}{2}\right)$.