

03/08/2018

(1)

Recap

Theorem 7, Chapter 5

Geometric multiplicity \leq Algebraic multiplicity

A is diagonalizable if and only if

geometric multiplicity = algebraic multiplicity

for every eigenvalue.

Inner product (Chapter 6)

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} \equiv \vec{u}^T \vec{v} = u_1 v_1 + \dots + u_n v_n$$

Norm

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \dots + v_n^2}$$

Distance

$$\text{dist}(\vec{u}, \vec{v}) \equiv \|\vec{u} - \vec{v}\|$$

Orthogonal vectors

If $\vec{u} \cdot \vec{v} = 0$, we say $\vec{u} \perp \vec{v}$

(2)

Pythagorean Theorem (Theorem 2)

$$\vec{u} \perp \vec{v} \text{ if and only if } \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Orthogonal complement

Let W be a subspace of \mathbb{R}^n

If $\vec{z} \perp$ all vectors in W , we say $\vec{z} \perp W$

$$W^\perp = \{ \text{all } \vec{z} \text{ satisfying } \vec{z} \perp w \}$$

Properties of W^\perp

*) W^\perp is a subspace.

*) Let $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$

$\vec{x} \perp W$ if and only if

$$\vec{x} \perp \vec{v}_1, \vec{x} \perp \vec{v}_2, \dots, \vec{x} \perp \vec{v}_p$$

Theorem 3, $(\text{Row } A)^\perp = \text{Nul } A$ $A: m \times n$
 $(\text{Col } A)^\perp = \text{Nul } A^T$

Dimensions of W and W^\perp

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W : a subspace of \mathbb{R}^n

let $\{\vec{v}_1, \dots, \vec{v}_p\}$ be a basis for W .

$$\Rightarrow W = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$$

$$\text{let } A = [\vec{v}_1 \cdots \vec{v}_p] \quad A = n \times p$$

$$W = \text{Col } A$$

$$\dim W = \dim \text{Col } A = \underbrace{\text{rank } A}_{\text{rank } A^T}$$

$$\dim W^\perp = \dim (\text{Col } A)^\perp = \dim \text{Nul } A^T$$

Recall the rank theorem (Theorem 14, Chapter 2)

$$B: m \times n$$

$$\boxed{\text{rank } B + \dim \text{Nul } B = n}$$

$$\boxed{B\vec{x} = \vec{0}}$$

Apply it to A^T

$$A^T = p \times n$$

$$\boxed{\text{rank } A^T + \dim \text{Nul } A^T = n}$$

$$\boxed{\dim W + \dim W^\perp = n}$$

Statement: Let W be a subspace in \mathbb{R}^n . (4)

We have

$$\boxed{\dim W + \dim W^\perp = n}$$

Geometric meaning:

Consider \mathbb{R}^3

$$W = \text{Span}\{\vec{v}_1\} \rightarrow \text{line}$$

$$W^\perp = \text{plane perpendicular to } \vec{v}_1$$

$$\dim W = 1, \dim W^\perp = 2$$

Ex. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & -1 \end{bmatrix}$

a) Find $\dim (\text{Col } A)^\perp$

b). Find a basis for $(\text{Col } A)^\perp$

a) $\text{Col } A$ is a subspace in \mathbb{R}^3

$\dim \text{Col } A = 2$

$\Rightarrow \dim (\text{Col } A)^\perp = 3 - \dim \text{Col } A = 1$

b). $(\text{Col } A)^\perp = \text{Nul } A^T$

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

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Add $(-1) \times R1$ to $R2$

$$\left\langle \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$$

Add $(-2) \times R2$ to $R1$

$$\left\langle \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{REF} \right\rangle$$

Basic: x_1, x_3 .

Free: x_2

Solution set

$$\begin{bmatrix} x \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\text{Nul } A^T = (\text{Col } A)^\perp$

Orthogonal set

Def: $\{\vec{u}_1, \dots, \vec{u}_p\}$ is called an orthogonal set

if $\vec{u}_i \cdot \vec{u}_j = 0$ for $i \neq j$

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Theorem 4, chapter 6

An orthogonal set of non-zero vectors
is linearly independent.

Proof Suppose $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthogonal set of non-zero vectors.

$$\text{Suppose } c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_p \vec{u}_p = \vec{0}$$

$$c_1 \vec{u}_j \cdot \vec{u}_1 + c_2 \vec{u}_j \cdot \vec{u}_2 + \dots + c_p \vec{u}_j \cdot \vec{u}_p = 0.$$

$$\Rightarrow \underbrace{c_j \vec{u}_j \cdot \vec{u}_j}_{=0} = 0.$$

$$\Rightarrow c_j = 0$$

Def. If a basis for W is an orthogonal set
then it is called an orthogonal basis for W

Ex. $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for \mathbb{R}^3

We need to show

- * 1) It is an orthogonal set ✓
- * 2) It is a basis for \mathbb{R}^3

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$$D. \vec{u}_1 \cdot \vec{u}_2 = 1 + 1 - 2 = 0$$

$$\vec{u}_1 \cdot \vec{u}_3 = -1 + 1 + 0 = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = -1 + 1 + 0 = 0$$

②. $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set

③ $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly independent.

Recall the basis theorem (Theorem 15, chapter 2).

$$\left\{ \begin{array}{l} \# \text{ of vectors in the set} = 3 \\ \dim \mathbb{R}^3 = 3 \end{array} \right.$$

Linear independence

$\Rightarrow \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a basis for \mathbb{R}^3 .

Conclusion: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis.

Statement: An orthogonal set of n non-zero vectors.

in \mathbb{R}^n is an orthogonal basis for \mathbb{R}^n .

Theorem 5: Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal basis for W .

Each \vec{y} in W can be represented as.

$$\vec{y} = c_1 \vec{u}_1 + \dots + c_p \vec{u}_p$$

The coefficients $\{c_j\}$ are given by.

$$c_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}$$

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Proof.

$$\vec{y} = c_1 \vec{u}_1 + \cdots + c_p \vec{u}_p$$

$$\vec{y} \cdot \vec{u}_j = c_1 \vec{u}_1 \cdot \vec{u}_j + \cdots + c_p \vec{u}_p \cdot \vec{u}_j$$

$$\vec{y} \cdot \vec{u}_j = c_j (\vec{u}_j \cdot \vec{u}_j)$$

$$\Rightarrow c_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}$$

$$\text{Ex. } \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

Find the coefficients in $\vec{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3$

$$\vec{y} \cdot \vec{u}_1 = 6 + 2 + 10 = 18$$

$$\vec{u}_1 \cdot \vec{u}_1 = 1 + 1 + 4 = 6$$

$$c_1 = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{18}{6} = 3$$

$$\vec{y} \cdot \vec{u}_2 = 6 + 2 - 5 = 3$$

$$\vec{u}_2 \cdot \vec{u}_2 = 1 + 1 + 1 = 3$$

$$c_2 = \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{3}{3} = 1$$

$$\vec{y} \cdot \vec{u}_3 = -6 + 2 = -4$$

$$\vec{u}_3 \cdot \vec{u}_3 = 1 + 1 = 2$$

$$c_3 = \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{-4}{2} = -2$$

$$\Rightarrow \vec{y} = 3 \vec{u}_1 + \vec{u}_2 - 2 \vec{u}_3$$

Orthogonal projection

⑨

Theorem 8

(the orthogonal decomposition theorem)

Let W be a subspace of \mathbb{R}^n

let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal basis for W .

a) Each \vec{y} in \mathbb{R}^n can be decomposed as

$$\vec{y} = \vec{g} + \vec{z}$$

where \vec{g} is in W and \vec{z} is in W^\perp

One solution for \vec{g} is given by.

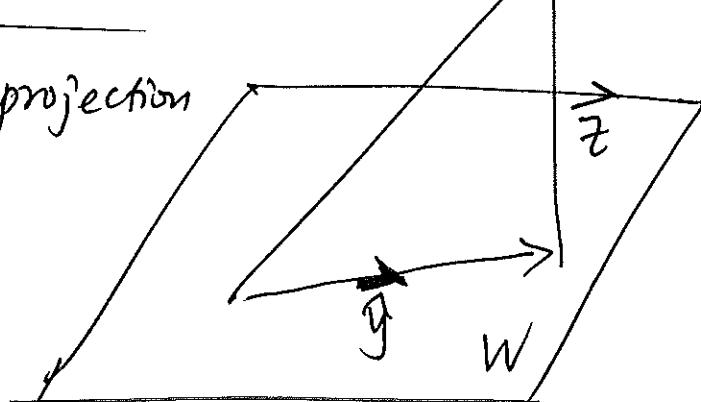
$$\vec{g} = \left(\frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \right) \vec{u}_1 + \dots + \left(\frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \right) \vec{u}_p$$

b) \vec{g} is unique.

\vec{g} is called the orthogonal projection

of \vec{y} onto W

Notation $\text{proj}_{W^\perp} \vec{y}$


projection of \vec{y} onto W .

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(a) We first show that \vec{y} is in W .

$$\vec{y} = \left(\frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \right) \vec{u}_1 + \cdots + \left(\frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \right) \vec{u}_p$$

$\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p$ are in W .

$\Rightarrow \vec{y}$ is in W .

Next we show $z = \vec{y} - \vec{g}$ is in W^\perp

We only need to show $(\vec{y} - \vec{g}) \perp \vec{u}_j$, $j=1, 2, \dots, p$.

$$(\vec{y} - \vec{g}) \cdot \vec{u}_j$$

$$= \vec{y} \cdot \vec{u}_j - \vec{g} \cdot \vec{u}_j$$

$$= \vec{y} \cdot \vec{u}_j - \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j} (\vec{u}_j \cdot \vec{u}_j)$$

$$= \vec{y} \cdot \vec{u}_j - \vec{y} \cdot \vec{u}_j = 0$$

$$\Rightarrow (\vec{y} - \vec{g}) \perp W.$$

$$\Rightarrow \vec{y} - \vec{g} \text{ is in } W^\perp$$

(b) Suppose we have two decompositions.

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$$\vec{y} = \vec{y}_1 + \vec{z}, \quad \vec{y}_1 \text{ is in } W \text{ and } \vec{z} \perp W$$

$$\vec{y} = \vec{y}_2 + \vec{z}_2, \quad \vec{y}_2 \text{ is in } W \text{ and } \vec{z}_2 \perp W$$

$$\Rightarrow \vec{y} + \vec{z} = \vec{y}_2 + \vec{z}_2$$

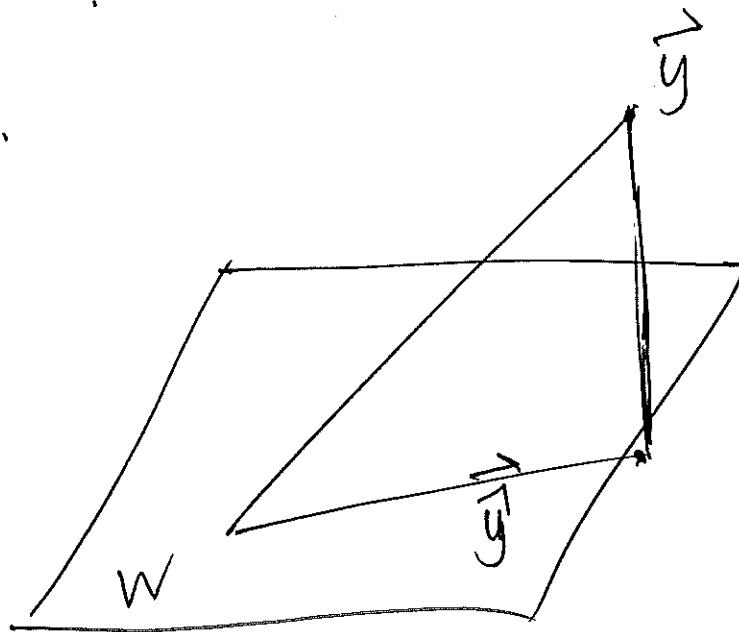
$$\Rightarrow \underbrace{(\vec{y} - \vec{y}_2)}_{\text{in } W} = \underbrace{\vec{z}_2 - \vec{z}}_{\text{in } W^\perp}$$

$$(\vec{y} - \vec{y}_2) \cdot (\vec{y} - \vec{y}_2) = (\vec{z}_2 - \vec{z}) \cdot (\underbrace{\vec{y} - \vec{y}_2}_{\text{in } W^\perp}) = 0$$

$$\Rightarrow \|\vec{y} - \vec{y}_2\|^2 = 0$$

$$\Rightarrow \vec{y} - \vec{y}_2 = \vec{0}$$

$$\Rightarrow \vec{y} = \vec{y}_2$$



Theorem 9 (Best representation theorem)

(12)

Let W be a subspace of \mathbb{R}^n .

Let \vec{y} be a vector in \mathbb{R}^n

We have $\|\vec{y} - \text{proj}_W \vec{y}\| < \|\vec{y} - \vec{v}\|$.

for all $\vec{v} \neq \text{proj}_W \vec{y}$ in W

Proof: Orthogonal decomposition of \vec{y}

$$\vec{y} = \vec{y}_w + \vec{z}$$

\vec{y}_w is in W , $\vec{z} \perp W$

$$\vec{y} - \vec{v} = (\vec{y}_w + \vec{z}) - \vec{v}$$

$$= (\vec{y}_w - \vec{v}) + \vec{z}$$

$$\|\vec{y} - \vec{v}\|^2 = \underbrace{\|\vec{y}_w - \vec{v}\|}_{\text{in } W}^2 + \|\vec{z}\|_{\text{in } W^\perp}^2$$

$$= \|(\vec{y}_w - \vec{v})\|^2 + \|\vec{z}\|^2$$

$$> \|\vec{z}\|^2 = \|\vec{y} - \vec{y}_w\|^2$$

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$$\|\vec{y} - \vec{v}\|^2 > \|\vec{y} - \vec{\hat{y}}\|^2$$

$$\Rightarrow \|\vec{y} - \vec{v}\| > \|\vec{y} - \text{proj}_w \vec{y}\|$$