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Recap

Q: How to find eigenvalues of A ?

Method: Solve $\det(A - \lambda I) = 0$
 λ : unknown

Q: How to find a basis for an eigenspace?

Method: Solve $(A - \lambda I)\vec{x} = \vec{0}$
 λ : known, \vec{x} : unknown

Similarity between two matrices

A and B are called similar if there exists P
such that $B = P^{-1}AP$

Theorem 4: A and B are similar

\Rightarrow They have the same characteristic polynomial
(same set of eigenvalues and algebraic multiplicities)

They also have the same geometric multiplicities

$\text{Nul}(A - \lambda I) \longleftrightarrow \text{Nul}(B - \lambda I)$
One-to-one
Correspondence

Diagonalization

Matrix A is called diagonalizable if there exists P
such that $P^{-1}AP = D$ (diagonal matrix)

Theorem 5:

$$A = n \times n$$

A is diagonalizable if and only if

A has n linearly independent eigenvectors.

Q: How to read out eigenvalues and eigenvectors from $P^{-1}AP = D$?

Method: $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ eigenvalues: $\lambda_1, \dots, \lambda_n$

$P = [\vec{v}_1 \dots \vec{v}_n]$ eigenvectors: $\vec{v}_1, \dots, \vec{v}_n$

Q: How to diagonalize A ?

Step 1: Find eigenvalues

Step 2: Find a basis for each eigenspace.

Step 3, step 4: Construct D and P

Theorem 6: $A = n \times n$

If A has n distinct eigenvalues,

then A is diagonalizable.

Ex. $A = \begin{bmatrix} 1 & 5 & 3 & 1 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

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Find out if A is diagonalizable.

A is triangular.

Eigenvalues = 1, 2, 3, 7

4 distinct eigenvalues

$\Rightarrow A$ is diagonalizable.

Q: What happens if A has fewer than n distinct eigenvalues?

$A = n \times n$

Theorem 7: $A = n \times n$

Characteristic polynomial.

$$\det(A - \lambda I) = C_n (\lambda - \xi_1) \cdots (\lambda - \xi_n)$$

$$= C_n (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_p)^{n_p}$$

$\lambda_1, \lambda_2, \dots, \lambda_p$ are p distinct eigenvalues.

Algebraic multiplicity of λ_k is n_k

$$n_1 + n_2 + \cdots + n_p = n$$

3 conclusions

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a) The geometric multiplicity of $\lambda_k \leq n_k$

b) A is diagonalizable if and only if
the geometric multiplicity of $\lambda_k = n_k$
for $k=1, 2, \dots, p$

c) Let B_k be a basis for the eigenspace of λ_k

The total collection $\{B_1, B_2, \dots, B_p\}$
is linearly independent.

Ex. $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

Try to diagonalize A

Step 1: Find eigenvalues

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

Co-factor expansion along R 1

$$= (2-\lambda) \det \begin{bmatrix} 3-\lambda & 4 \\ 0 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda)^2 (3-\lambda)$$

Eigenvalues 2 (algebraic multiplicity 2)

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Step 2: Find a basis for each eigenspace

$$\boxed{\lambda_1=2} \quad (A-\lambda_1 I) = A-2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \textcircled{1} & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{REF}$$

Basic variable: x_1

Free variables: x_2, x_3 .

Solution set:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

A basis: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\boxed{\lambda_2=3} \quad A-\lambda_2 I = A-3I = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

Row reduction: Add R_1 to R_2

$$\rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \textcircled{4} \\ 0 & 0 & -1 \end{bmatrix}$$

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Add $\frac{1}{4} \times R_2$ to R_3 \rightarrow $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{REF}$

Basic variables: x_1, x_3

Free variable: x_2

Solution set: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

A basis: $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Step 3, Step 4.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -4 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\boxed{P^{-1}AP = D}$$

Ex. $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$

Try to diagonalize A:

Step 1: find eigenvalues.

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$$\begin{aligned}\det(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 3-\lambda & 4 \\ 1 & 0 & 2-\lambda \end{bmatrix} \\ &= (2-\lambda) \det \begin{bmatrix} 3-\lambda & 4 \\ 0 & 2-\lambda \end{bmatrix} \\ &= (2-\lambda)^2 (3-\lambda)\end{aligned}$$

Eigenvalues: 2 (Algebraic multiplicity 2)
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Step 2 find a basis for each eigenspace.

$\lambda_1 = 2$ $A - \lambda_1 I = A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix}$

Row reduction.

Interchange R_1 and R_3

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Add $(-1) \times R_1$ to R_2

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{REF}$$

Basic variables: x_1, x_2

Free variable: x_3

$$l = \# \text{ of free variables}$$

$$= \dim \text{Nul}(A - \lambda_1 I)$$

$$= \text{geometric multiplicity of } \lambda_1$$

geometric multiplicity < algebraic multiplicity.

$\Rightarrow A$ is not diagonalizable.

Sec 6.1: Inner product, norm, orthogonality

Def: $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$\underbrace{\vec{u} \cdot \vec{v}}_{\substack{\text{Inner product} \\ \text{dot product}}} \equiv \underbrace{[u_1 \dots u_n]}_{1 \times n} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{n \times 1} = u_1 v_1 + \dots + u_n v_n$$

$$\vec{u}^T \cdot \vec{v}$$

$1 \times n \quad n \times 1$

Theorem 1 (properties of inner product)

a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

c) $(c \vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c \vec{v})$

d) $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$

proof of d).

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$$\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + \dots + u_n^2$$

Def (norm)

$$\underbrace{\|\vec{v}\|}_{\text{norm of } \vec{v}} \equiv \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

$$\text{A more useful form: } \|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$$

Properties of norm:

$$*) \|\vec{v}\| = 0 \quad \text{if and only if } \vec{v} = \vec{0}$$

$$*) \|\vec{c}\vec{v}\| = |c| \|\vec{v}\|$$

$$\begin{aligned} \text{proof: } \|\vec{c}\vec{v}\|^2 &= (\vec{c}\vec{v}) \cdot (\vec{c}\vec{v}) = c^2 (\vec{v} \cdot \vec{v}) \\ &= c^2 \|\vec{v}\|^2 \end{aligned}$$

$$\rightarrow \|\vec{c}\vec{v}\| = |c| \|\vec{v}\|$$

*) Consider $\vec{v} \neq \vec{0}$

$$\text{let } \vec{u} = \left(\frac{1}{\|\vec{v}\|}\right) \vec{v}$$

$$\|\vec{u}\| = \left\| \left(\frac{1}{\|\vec{v}\|} \right) \vec{v} \right\| = \frac{1}{\|\vec{v}\|} \|\vec{v}\| = 1$$

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This is called a unit vector.

Ex. $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Normalize \vec{v} :

$$\|\vec{v}\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

Def. (distance between two vectors).

$$\underbrace{\text{dist}(\vec{u}, \vec{v})}_{\substack{\text{distance between} \\ \vec{u} \text{ and } \vec{v}}} \equiv \|\vec{u} - \vec{v}\|.$$

Ex. $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Find the distance between \vec{u} and \vec{v}

$$\vec{u} - \vec{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$$

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$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

Def (orthogonal vectors).

If $\vec{u} \cdot \vec{v} = 0$, then we say \vec{u} and \vec{v} are orthogonal to each other.

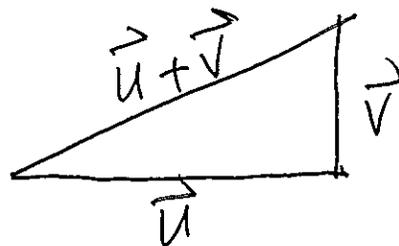
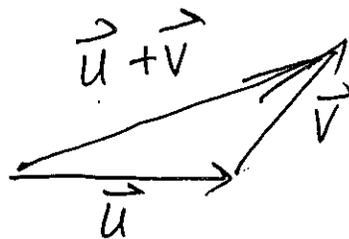
Notation: $\vec{u} \perp \vec{v}$

Theorem 2 (Pythagorean theorem)

$\vec{u} \perp \vec{v}$ if and only if $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

Geometric meaning:

Recall the geometric meaning of vector addition



proof $\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$

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$$= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u} \cdot \vec{v}$$

$$\vec{u} \cdot \vec{v} = 0 \text{ if and only if } \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

end of proof

Def: let W be a subspace of \mathbb{R}^n .

If $\vec{z} \perp$ all vectors in W

then we say $\vec{z} \perp W$

Def: (orthogonal complement of a subspace)

$$W^\perp = \{ \text{all } \vec{z} \text{ satisfying } \vec{z} \perp W \}$$

orthogonal
complement of W

properties of W^\perp

* W^\perp is a subspace

* Suppose $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$

\vec{x} is in W^\perp if and only if $\vec{x} \perp \vec{v}_1, \dots, \vec{x} \perp \vec{v}_p$

Relation between Nul A and Row A

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$$A = m \times n \quad A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

We write

$$A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix} \quad \vec{a}_i = [a_{i1} \dots a_{in}]$$

\vec{a}_i is in \mathbb{R}^n .

$$\Rightarrow \text{Row } A \text{ is in } \mathbb{R}^n$$

$$\underline{\text{Nul } A \text{ is in } \mathbb{R}^n}$$

\vec{x} is in Nul A

$$\iff A\vec{x} = \vec{0}$$

$$\iff \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix} \vec{x} = \vec{0}$$

Recall the row-vector rule for matrix-vector multiplication.

$$\iff \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \\ \vdots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix} = \vec{0}$$

$$\iff \vec{x} \perp \vec{a}_1, \quad \vec{x} \perp \vec{a}_2, \quad \dots, \quad \vec{x} \perp \vec{a}_m$$

$$\iff \vec{x} \perp (\text{Row } A)$$

$\iff \vec{x}$ is in $(\text{Row } A)^\perp$

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Conclusion:

$$\text{Nul } A = (\text{Row } A)^\perp$$

$$\text{Nul } A^T = (\text{Col } A)^\perp$$

Theorem 3: