AMS 10/10A, Homework 9

Problem 1. Let A and B be two similar matrices. Prove that A^k is also similar to B^k for positive integer k.

Problem 2. Matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$ has the diagonalization given below. Use Theorem 5 in Section 5.3 to find the eigenvalues and corresponding eigenvectors of A.

$$P = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Problem 3. Diagonalize each of the matrices below, if possible.

$$A = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} -5 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Problem 4. A is 9×9 matrix with three distinct eigenvalues. Two of the eigenvalues have geometric multiplicity 3; and one eigenvalue has geometric multiplicity 2. Is A diagonalizable? Why?

Problem 5. Is the following matrix diagonalizable? Why?

$$\begin{bmatrix} 1 & 2 & -2 & 0 & 1 & -3 & 3 \\ 0 & 2 & 1 & -2 & 3 & 0 & 2 \\ 0 & 0 & 3 & -1 & 9 & 11 & 2 \\ 0 & 0 & 0 & 4 & 7 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

Problem 6. Prove that if A is both diagonalizable and invertible, then A^{-1} is also diagonalizable and invertible.

Problem 7. Prove that if A is diagonalizable, then A^k is also diagonalizable for any positive integer k.

Problem 8. Let $u = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$. Compute the following quantities.

$$u^T v, \qquad v^T u, \qquad \left(\frac{u^T u}{v^T u}\right) u, \qquad \|u - v\|$$

Problem 9. Determine if the following pair of vectors are orthogonal.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2.5 \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 25 \\ -7 \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} 13 \\ -3 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Problem 10. Prove the parallelogram law:

$$\|u + v\|^2 + \|u - v\|^2 \quad = \quad 2\|u\|^2 + 2\|v\|^2$$

where u and v are vectors in $\mathbb{R}^n$.

Problem 11. Suppose a vector x is orthogonal to both vectors y and z. Prove that x is orthogonal to any vector in $span\{y, z\}$.

Problem 12. Let $H = Col(A)$, where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}$. Find $H^{\perp}$, the orthogonal complement of H.

Problem 13. Let $u = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, and $H = span\{u\}$. What is the dimension of $H^{\perp}$, the orthogonal complement of H.

Problem 14. Let A be a 7×5 matrix. What is the smallest possible dimension of $[Col(A)]^{\perp}$? Explain your answer.

Problem 15. Determine if the following sets of vectors are orthogonal.

$$\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} \right\}, \qquad \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} \right\}$$

Problem 16. Let $u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, and $u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

- Show that $\{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^3$;
- Express $x = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ as a linear combination of $\{u_1, u_2, u_3\}$.

Problem 17.

Let $u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $u_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$, and $v = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix}$.

It is known that $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for $\mathbb{R}^4$. Write v as the sum of two vectors, one in $span\{u_1, u_2\}$ and the other in $span\{u_3, u_4\}$.