3/1/2018

Recap:

Eigenspace $= Nul(A - \lambda I)$

Characteristic equation

$$\det(A - \lambda I) = 0$$

Characteristic polynomial

$$\det(A - \lambda I), \quad \text{degree } n, \quad A = n \times n$$

$$\downarrow$$

$$(\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots$$

Algebraic multiplicity of λ_1: n_1

Geometric multiplicity of λ_1: $\dim Nul(A - \lambda_1 I)$
$= \#$ of free variables in $(A - \lambda_1 I)\vec{x} = \vec{0}$

Geometric multiplicity $\neq$ Algebraic multiplicity

Geometric multiplicity is based on eigenspace

Algebraic multiplicity is based on characteristic polynomial

Theorem 1, chapter 5

Eigenvalues of $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 \cdots & 0 & a_{nn} \end{bmatrix}$

are $a_{11}, a_{22}, \cdots, a_{nn}$

Theorem 2, chapter 5

Eigenvectors for distinct eigenvalues are linearly independent.

Q: How to find eigenvalues?

Method: Solve $\det(A - \lambda I) = 0$.

λ is unknown

Q: How to find a basis for the eigenspace of eigenvalue λ?

Method: Solve $(A - \lambda I)\vec{x} = 0$.

$\vec{x}$ is unknown, λ is known

Caution: Row reduction is not a tool for finding eigenvalues!

$$A \xrightarrow{\text{row reduction}} B$$

A and B have different sets of eigenvalues.

Ex $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Find eigenvalues and find a basis for each eigenspace.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)(-1)^{3+3} \det \begin{bmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)^2 - (-2)^2\right]$$

$$= (1-\lambda)\underbrace{\left[\lambda^2 - 2\lambda - 3\right]}_{(\lambda-3)(\lambda+1)}$$

Characteristic eq:

$$(\lambda-1)(\lambda-3)(\lambda+1) = 0$$

eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = -1$

$\boxed{\text{A basis for eigenspace of } \lambda_1 = 1}$

$$A - \lambda_1 I = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{Interchange R1 and R2}} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{(-\frac{1}{2})\times R1,\ -\frac{1}{2}\times R2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{REF}$$

Basic variables = x_1, x_2

Free variable : x_3.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of $\lambda_1 = 1$.

$\boxed{\text{A basis for the eigenspace of } \lambda_2 = 3}$

$$A - \lambda_2 I = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\text{Add } (-1)\times R1 \text{ to } R2} \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\xrightarrow{\text{Interchange } R2 \text{ and } R3} \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{(-\frac{1}{2})\times R_1,\ (\frac{-1}{2})\times R2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{REF}$$

Basic variables: x_1, x_3

Free variable: x_2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the eigenspace of $\lambda_2 = 3$

A basis for the eigenspace of $\lambda_3 = -1$

$$A - \lambda_3 I = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the eigenspace of $\lambda_3 = -1$

Complex eigenvalues.

<u>Ex</u> $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Find eigenvalues and find a basis for each eigenspace.

$$\det(A-\lambda I)=\det\begin{bmatrix}-\lambda & 2\\ -2 & -\lambda\end{bmatrix}$$

$$=(-\lambda)(-\lambda)-(-2)2=\lambda^2+4$$

Characteristic eq: $\lambda^2+4=0$

$$\lambda^2-(-4)=0$$

$$\lambda^2-(2i)^2=0$$

$$(\lambda-2i)(\lambda+2i)=0$$

Eigenvalues: $\lambda_1=2i$, $\lambda_2=-2i$

A basis for the eigenspace of $\lambda_1=2i$

$$A-\lambda_1 I=\begin{bmatrix}-2i & 2\\ -2 & -2i\end{bmatrix}$$

$$\xrightarrow{\text{Add } (+i)\times R1 \text{ to } R2}\begin{bmatrix}-2i & 2\\ 0 & 0\end{bmatrix}$$

$$\xrightarrow{(\frac{1}{2}i)\times R1}\begin{bmatrix}1 & i\\ 0 & 0\end{bmatrix}\leftarrow \text{REF}$$

Basic variable: x_1,

Free variable: x_2

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-i\,x_2\\ x_2\end{bmatrix}=x_2\begin{bmatrix}-i\\ 1\end{bmatrix}$$

$\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of $\lambda_1 = 2i$

A basis for the eigenspace of $\lambda_2 = -2i$

$$A - \lambda_2 I = \begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix}$$

$$\xrightarrow{\text{Add } (-i) \times R_1 \text{ to } R_2} \begin{bmatrix} 2i & 2 \\ 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\left(-\frac{1}{2}i\right) \times R1} \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \leftarrow \text{REF}$$

$\vdots$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of $\lambda_2 = -2i$

Similarity between two matrices.

Def: We say A is similar to B if there is an invertible matrix P such that

$$B = P^{-1} A P$$

Statement: If A is similar to B,

then B is similar to A.

proof: We need to find an invertible matrix Q such that

$$A = Q^{-1} B Q$$

We are given $B = P^{-1} A P$.

$$P B P^{-1} = P(P^{-1} A P) P^{-1}$$

$$(P^{-1})^{-1} B P^{-1} = (P P^{-1}) A (P P^{-1})$$

$$(P^{-1})^{-1} B P^{-1} = A$$

$$\Rightarrow \quad A = Q^{-1} B Q, \qquad Q = P^{-1}$$

Def: The mapping $A \longrightarrow P^{-1} A P$ is called a similarity transformation.

Q: Why study similarity transformation?

Theorem 4: If A and B are similar, then they have the same characteristic polynomial.

- $\rightarrow$ Same set of eigenvalues.
- $\rightarrow$ Same algebraic multiplicities.

proof: $B = P^{-1}AP$

$$\det(B-\lambda I) = \det(P^{-1}AP-\lambda I)$$
$$= \det(P^{-1}AP-\lambda P^{-1}IP)$$
$$= \det[P^{-1}(A-\lambda I)P]$$
$$= \det(P^{-1})\,\underline{\det(A-\lambda I)}\,\det P\,.$$
$$= \det(A-\lambda I)$$

Statement: If A and B are similar, then they also have the same geometric multiplicities.

proof: Suppose λ is an eigenvalue of A, and B.
Suppose $\vec{z}$ is in the eigenspace of B for eigenvalue λ.

$$(B-\lambda I)\vec{z} = \vec{0}. \qquad B = P^{-1}AP$$
$$(P^{-1}AP-\lambda I)\vec{z} = \vec{0}$$
$$(P^{-1}AP-\lambda P^{-1}IP)\vec{z} = \vec{0}$$
$$P^{-1}\underline{(A-\lambda I)(P\vec{z})} = \vec{0}$$
$$(A-\lambda I)\underline{(P\vec{z})} = \vec{0}.$$

$\Rightarrow$ $P\vec{z}$ is in the eigenspace of A for eigenvalue λ.

$$\text{Nul}(A-\lambda I) \xleftrightarrow[\text{correspondence}]{\text{one-to-one}} \text{Nul}(B-\lambda I)$$

$$\dim \text{Nul}(A-\lambda I) = \dim \text{Nul}(B-\lambda I)$$

Sec 5.3. Diagonalization.

Def: A diagonal matrix = $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$

Def: We say matrix A is diagonalizable if there is an invertible matrix P such that

$P^{-1}AP$ is diagonal.

Q: Why study diagonalization?

Theorem 5: The diagonalization theorem

$A = n \times n$.

A is diagonalizable if and only if A has n linearly independent eigenvectors.

---

proof: $\boxed{\text{"Only if" part.}}$

A is diagonalizable $\longrightarrow$ A has n linearly independent eigenvectors.

Suppose $P^{-1}AP = D$, $\quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\Rightarrow \quad AP = PD$

Let $P = [\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_n]$

$\Rightarrow A[\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_n] = [\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Focus on the j-th column

$$A\vec{v}_j = [\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_n] \begin{bmatrix} 0 \\ \vdots \\ \lambda_j \\ \vdots \\ 0 \end{bmatrix} = \lambda_j \vec{v}_j$$

$$\boxed{A\vec{v}_j = \lambda_j \vec{v}_j}$$

λ_j is an eigenvalue and $\vec{v}_j$ is an eigenvectors.

$\Rightarrow \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ are a set of n linearly independent eigenvectors.

"If" part

A has n linearly independent eigenvectors $\longrightarrow$ A is diagonalizable.

Suppose $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ are n linearly independent eigenvectors.

Let $P = [\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_n]$

Linear independence $\Longrightarrow$ P is invertible.

$$A\vec{v}_j = \lambda_j \vec{v}_j$$

$$\Rightarrow A\underbrace{[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]}_{P} = \underbrace{[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]}_{P} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Q: How to read out eigenvalues and eigenvectors for the given diagonalization.

$$P^{-1}AP = D$$

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ $\quad\lambda_1, \cdots \lambda_n$ are eigenvalues.

$P = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]$ $\quad\vec{v}_1, \cdots \vec{v}_n$ are eigenvectors.

Q: How to construct the diagonalization of A.

Step 1: Find all eigenvalues of A.

Step 2: Find a basis for each eigenspace.

Step 3: Construct $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Step 4: Construct $P = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]$

Ex. $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Eigenvalues: $1, \ 3, \ -1$

Eigenvectors: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$D = \begin{bmatrix} 1 & & 0 \\ & 3 & \\ 0 & & -1 \end{bmatrix}$$

$$P = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\boxed{P^{-1}AP = D.}$$

Theorem 6: $A: n \times n$.

Suppose A has n distinct eigenvalues.

Then A is diagonalizable.

Relation between finding eigenvalues of A and diagonalizing A

In theoretical discussion.

$A \xrightarrow{\text{Exact calculation.}}$ Exact eigenvalues and eigenvectors.

$\longrightarrow$ Diagonalizing A.

In numerical computation.

$A \xrightarrow{\text{Numerical computation.}}$ Diagonalizing A approximately.

$\longrightarrow$ Read out eigenvalues and eigenvectors approximately.