2/27/2018

Recap

Determinants

Theorem 4, chap 3

$\det A \neq 0$ if and only if A is invertible.

Theorem 5, chap 3

$$\det(A^T) = \det A$$

Theorem 6, chap 3

$$\det(AB) = (\det A)(\det B)$$

$$\det(A^K) = (\det A)^K$$

$$\det(A^{-1}) = \frac{1}{\det A}$$

Other properties of determinants

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 \cdots 0 & & a_{nn} \end{bmatrix} = a_{11} \cdots a_{nn}$$

$\det(\alpha A) = \alpha^n \det A$, $\quad A: n \times n$

$\det(-A) = (-1)^n \det A.$

Caution:

In general, $\det(A+B) \neq \det A + \det B$

Eigenvalue problem

*) $\vec{x}$ is an eigenvector of A

$\iff A\vec{x} = \lambda\vec{x}$ for some scalar λ
(and $\vec{x} \neq \vec{0}$)

*) λ is an eigenvalue of A

$\iff A\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$

$\iff (A - \lambda I)\vec{x} = \vec{0}$ has a non-trivial solution.

$\iff (A - \lambda I)$ is NOT invertible

$\iff \det(A - \lambda I) = 0$.

Previously.

Q: Given $\vec{u}$, how to verify if $\vec{u}$ is an eigenvalue of A?

Now we look at

Q: Given eigenvalue λ,
how to find an eigenvector corresponding to λ?

Ex $A = \begin{bmatrix} 6 & 2 \\ -2 & 1 \end{bmatrix}$

1) verify that 5 is an eigenvalue.

2). Find a corresponding eigenvector.

part 1): $\det(A-5I) = \det\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$

$= 1\times(-4)-(2)\times(-2) = -4+4=0$. ✓

$\Longrightarrow$ 5 is an eigenvalue of A.

part 2) An eigenvector $\vec{x}$ satisfies.

$$A\vec{x} = 5\vec{x}$$

$$\Longleftrightarrow (A-5I)\vec{x} = \vec{0}$$

We solve $(A-5I)\vec{x}$ for a non-trivial solution.

Row reduction

$$A - 5I = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$$

Add $2 \times R1$ to $R2$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \leftarrow \text{reduced echelon form}$$

Basic variable: x_1

Free variable: x_2

Solution set $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

An eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Def: (eigenspace)

Suppose λ is an eigenvalue of A.

$\text{Nul}(A - \lambda I)$ is called the eigenspace corresponding to eigenvalue λ.

Back to example:

The eigenspace corresponding to eigenvalue 5

$$= \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

Ex. $A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 9 & 2 \\ 1 & 3 & 4 \end{bmatrix}$

1). Verify that 3 is an eigenvalue.

2). Find a basis for the eigenspace corresponding to eigenvalue 3.

part 1): $\det(A-3I) = \det\begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ 1 & 3 & 1 \end{bmatrix} = 0$

$\Rightarrow$ 3 is an eigenvalue of A

part 2): Row reduction

REF

$$A - 3I = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\text{Add } (-1)\times R1 \text{ to } R3]{\text{Add } (-2)\times R2 \text{ to } R2} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basic variable: x_1

Free variables: x_2, x_3

Solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to eigenvalue 3.

__Summary__:

__Q__: Given eigenvalue λ, how to find a basis for the eigenspace?

__Method__: Solve $(A - \lambda I)\vec{x} = 0$.

Write out the solution set.

__Q__: How to find eigenvalues?

$$\boxed{\lambda \text{ is an eigenvalue} \iff \det(A - \lambda I) = 0.}$$

__Method__: We solve $\det(A - \lambda I) = 0$.

Def: (Characteristic equation)

$\det(A - \lambda I) = 0$ is called the characteristic equation of matrix A.

A special case:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ \vdots & \ddots & \\ 0 & \cdots\ 0 & a_{nn} \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{bmatrix} a_{11} - \lambda & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ \vdots & & \vdots \\ 0 & \cdots\ 0 & a_{nn} - \lambda \end{bmatrix}$$

$$= (a_{11} - \lambda) \cdots (a_{nn} - \lambda)$$

Characteristic equation.

$$(a_{11} - \lambda) \cdots (a_{nn} - \lambda) = 0.$$

Solutions: $\underbrace{a_{11}, \cdots a_{nn}}_{\text{eigenvalues}}$

$\boxed{\text{Theorem 1, chapter 5}}$

The eigenvalues of $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ \vdots & \ddots\ 0 & a_{nn} \end{bmatrix}$ are $a_{11}, \cdots, a_{nn}$

Ex. $A = \begin{bmatrix} 2 & 7 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ Eigenvalues are 2, 0, 1

Ex $A = \begin{bmatrix} 2 & 3 \\ -2 & -5 \end{bmatrix}$

Goal: To solve for eigenvalues.

$$\det(A-\lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ -2 & -5-\lambda \end{bmatrix}$$

$$= (2-\lambda)(-5-\lambda) - 3(-2)$$

$$= \lambda^2+3\lambda-4 = (\lambda+4)(\lambda-1)$$

Characteristic equation:

$$(\lambda+4)(\lambda-1)=0.$$

Eigenvalues: -4, 1

Caution: Row reduction is not a tool for Calculating eigenvalues.!

$$A \xrightarrow{\text{row reduction}} B$$

In general, A and B have different sets of eigenvalues.

Ex.
$$\underbrace{\begin{bmatrix} 2 & 3 \\ -2 & -5 \end{bmatrix}}_{A} \xrightarrow{\text{Add R1 to R2}} \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}}_{B}$$

eigenvalues: $-4, 1$ eigenvalues $2, -2$

Next we introduce 3 items.

*) Characteristic polynomial $\det(A-\lambda I)$

*) Algebraic multiplicity of an eigenvalue

*) Geometric multiplicity of an eigenvalue.

Statement 1: Suppose A: $n \times n$.

$$\boxed{\text{Then } \det(A-\lambda I) \text{ is a polynomial of degree } n}$$

proof: $\det A = \sum\limits_{(j_1, j_2, \cdots, j_n)} \varepsilon(j_1, \cdots j_n) a_{1j_1} a_{2j_2} \cdots a_{nj_n}$.

$$(A-\lambda I) = \begin{bmatrix} a_{11}-\lambda & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = \underbrace{(a_{11}-\lambda)\cdots(a_{nn}-\lambda)}_{\text{polynomial of degree } n.} + \underbrace{\cdots}_{\substack{\text{polynomial of} \\ \text{degree} < n.}}$$

Conclusion: $\det(A-\lambda I)$ = polynomial of degree n.
is called the characteristic polynomial.

Statement 2:

Recall the fundamental theorem of algebra.

$$\det(A-\lambda I) = c(\lambda-\xi_1)\cdots(\lambda-\xi_n)$$

We write it as

$$\det(A-\lambda I) = c(\lambda-\lambda_1)^{n_1}\cdots(\lambda-\lambda_k)^{n_k}.$$

$\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalues of A.

Def. (Algebraic multiplicity of an eigenvalue).

Algebraic multiplicity of eigenvalue λ_j is defined as n_j

Def (Geometric multiplicity of an eigenvalue)

Geometric multiplicity of λ_j

$$\begin{aligned} &\equiv \dim(\text{eigenspace of } \lambda_j) \\ &= \dim(\mathrm{Nul}(A-\lambda_j I)) \\ &= \text{\# of free variables in } (A-\lambda_j I)\vec{x}=\vec{0} \end{aligned}$$

Ex. $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}$$

$$= (3-\lambda)^2$$

3 is an eigenvalue.

$\boxed{\text{Algebraic multiplicity of 3}}$ is 2

$$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Basic variable : x_2

Free variable : x_1

Solution set of $(A-3I)\vec{x} = \vec{0}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

eigenspace of 3 $= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\boxed{\text{Geometric multiplicity} = 1}$

In general, $\boxed{\text{geometric multiplicity} \neq \text{algebraic multiplicity}}$

The invertible matrix theorem.

a) A is invertible

⋮

r) $\dim \operatorname{Nul} A = 0$.

s). 0 is Not an eigenvalue of A

s) is equivalent to t)

t) $\det A \neq 0$.

t) is equivalent to a)

Theorem 2:

Suppose $\lambda_1, \lambda_2, \cdots, \lambda_r$ are distinct eigenvalues of A.

Let $\vec{v}_j$ be an eigenvector corresponding to λ_j

Then $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r\}$ is linearly independent.

Proof: (We use proof by contradiction)

Suppose $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r\}$ is linearly dependent.

$\Rightarrow c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r = \vec{0}$

$(c_1, c_2, \cdots, c_r)$ are not all zero.

$\Rightarrow$ $(c_1, c_2, \cdots, c_{r-1})$ are not all zero.

If c_r is the only non-zero,

$\Rightarrow \quad c_r \vec{v}_r = 0$.

$\Rightarrow \quad \vec{v}_r = 0$. impossible!

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r = \vec{0}$$

Multiply by A. $\quad A\vec{v}_j = \lambda_j \vec{v}_j$

$$\hookrightarrow c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_r\lambda_r\vec{v}_r = 0.$$

Multiply by λ_r

$$\hookrightarrow c_1\lambda_r\vec{v}_1 + c_2\lambda_r\vec{v}_2 + \cdots + c_r\lambda_r\vec{v}_r = 0$$

$$\underbrace{c_1(\lambda_1-\lambda_r)}\vec{v}_1 + \underbrace{c_2(\lambda_2-\lambda_r)}\vec{v}_2 + \cdots + \underbrace{c_{r-1}(\lambda_{r-1}-\lambda_r)}\vec{v}_{r-1} = 0$$

$c_1(\lambda_1-\lambda_r)$, $c_2(\lambda_2-\lambda_r)$, $\cdots$, $c_{r-1}(\lambda_{r-1}-\lambda_r)$ are not all zero

$\Rightarrow$. $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_{r-1}\}$ is linearly dependent.

Repeat the process.

$\Rightarrow \{\vec{v}_1\}$ is linearly dependent.

$\Rightarrow \vec{v}_1 = \vec{0}$, contradiction!

Conclusion:

$\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r\}$ is linearly independent.

end of proof: