AMS 10/10A, Homework 8

Problem 1. Determine whether v is an eigenvector of matrix A given below.

$$A = \begin{bmatrix} 3 & -5 & -1 \\ 1 & 9 & 1 \\ 1 & 5 & 5 \end{bmatrix}, \qquad v = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

Problem 2. Determine if $\lambda = 1$ is an eigenvalue of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ -2 & -3 & -4 \end{bmatrix}$.

Problem 3. Find the eigenvalues and associated eigenvectors for each of the matrices given below.

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Problem 4. Let a, b and c be arbitrary real numbers. Prove that the eigenvalues of the matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

cannot be complex numbers.

Problem 5. Let θ be an arbitrary angle such that $\cos(\theta) \neq 0$. Find the eigenvalues and associated eigenvectors of the following matrix.

$$A = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{bmatrix}$$

Problem 6. Let A be an $n \times n$ matrix and λ be an eigenvalue of A. Prove that λ^k is an eigenvalue of A^k, where k is a positive integer.

Problem 7. Let λ be an eigenvalue of the invertible matrix A. Prove that λ^{-1} is an eigenvalue of A^{-1}.

Problem 8. Let A be an $n \times n$ matrix. Prove that A and A^T have the same eigenvalues.
Hint: Consider the characteristic equation.

Problem 9. Consider an $n \times n$ matrix A with the property that the sum of all columns is a vector with all entries being the same number, as shown below.

$$a_1 + a_2 + \cdots + a_n = \begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix},$$

where a_i is the ith column of A. Prove that s is an eigenvalue of A.
Hint: Try to find an eigenvector corresponding to that eigenvalue.

Problem 10. Let v_1 and v_2 be two eigenvectors of a square matrix A corresponding to the same eigenvalue λ. Prove that any linear combination of v_1 and v_2, if not equal to the zero vector, is also an eigenvector of A.

Problem 11. Find all eigenvalues of the following matrix. For each eigenvalue, find its algebraic multiplicity and geometric multiplicity.

$$A = \begin{bmatrix} 3 & 2 & -4 & 1 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 12. Find the value(s) of α in matrix A such that the geometric multiplicity of $\lambda = 4$ is 2.

$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & \alpha & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Problem 13. Find the eigenvalues and associated eigenvectors for the following matrix.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & -2 \\ 0 & 2 & 8 \end{bmatrix}$$

Problem 14. Consider a general 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let λ_1 and λ_2 be the two eigenvalues of matrix A. Prove that $\lambda_1 \cdot \lambda_2 = det(A)$.
Hint: Write out the characteristic equation.

Problem 15. Consider a general 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let λ_1 and λ_2 be the two eigenvalues of matrix A. Prove that $\lambda_1 + \lambda_2 = a + d$.
Hint: Write out the characteristic equation.