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Recap

Row vectors

Row A

$A \xrightarrow{\text{row reduction}} B$ (echelon form)

Non-zero rows of B form a basis for Row B = Row A

$$\underset{\text{rank } A}{\overset{\|}{\#\text{ of pivot positions}}} = \text{Dim Row } A = \underset{\text{rank } A^T}{\overset{\|}{\text{Dim Col } A^T}}$$

$$\boxed{\text{rank } A = \text{rank } A^T}$$

Determinants

Co-factor expansion along row i

$$\det A = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det A_{ij}$$

Co-factor expansion along column j

$$\det A = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det A_{ij}$$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & a_{22} & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots\ 0 & a_{nn} \end{bmatrix} = a_{11} a_{22} \cdots a_{nn}$$

procedure of calculating det A using EROs

*) Row reduce A to $B_1, \to B_2, \to B_3, \cdots, \to B_N$

echelon form

*) Include the effect of each ERO

$$A \xrightarrow{\text{Add } c\times(\text{row } k) \text{ to } (\text{row } l)} B$$

$$\det A = \det B$$

$$A \xrightarrow{\text{Interchange two rows}} B$$

$$\det A = -\det B$$

$$A \xrightarrow{\text{Multiply } (\text{row } k) \text{ by } r \neq 0} B$$

$$\det A = \frac{1}{r} \det B$$

Ex. $A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 6 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ multiply R1 by 1/2

$$\det A = 2 \det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$ Add $(-3)\times$R1 to R2, Add $(-2)\times$R1 to R3

$$= 2 \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & -3 & 1 \end{bmatrix}$$ Interchange R2 and R3

$$= -2 \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= -2 \cdot 1 \cdot (-3)(-1) = -6$$

Theoretical discussion: connection between determinant and invertibility

A: $n \times n$

$$A \xrightarrow{\text{row reduction}} B \quad \text{(echelon form)}$$

$$\det A = \underbrace{(-1)^r k_1 \cdots k_q}_{\text{non-zero}} \det B$$

$\Rightarrow \det A \neq 0$ if and only if $\det B \neq 0$

If # of pivot positions $< n$

$\Rightarrow$ Last row of B is all zeros.

$\Rightarrow \det B = 0$ (co-factor expansion along the last row)

$\Rightarrow \det A = 0$

If # of pivot positions $= n$

$$B = \begin{bmatrix} b_{11} & & b_{1n} \\ 0 & \ddots & \\ \vdots & \ddots & \ddots \\ 0 & \cdots\ 0 & b_{nn} \end{bmatrix}$$

$\det B = b_{11} \cdots b_{nn} \neq 0$.

$\Rightarrow \det A \neq 0$

Statement: A : $n \times n$.

$\det A \neq 0$ if and only if A has n pivot positions

Theorem 4 (chap 3)

$\det A \neq 0$ if and only if A is invertible.

Statement: $\det A \neq 0$ if and only if columns of A are linearly independent.

Special case:

Suppose one column of A is a multiple of another column. $\Rightarrow$ ~~det A ≠ 0~~ $\det A = 0$

Statement: $\det A \neq 0$ if and only if rows of A are linearly independent.

Special case:

Suppose one row of A is a multiple of another row $\Rightarrow$ $\det A = 0$

Ex

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 6 & -4 \\ 2 & -4 & 5 \end{bmatrix}$$

$$\text{col } 2 = (-2)(\text{col } 1)$$

$$\Rightarrow \det A = 0$$

Q: How is $\det A$ related to $\det A^T$?

Theorem 5 (Chap 3)

$$\det A^T = \det A$$

We prove it using mathematical induction.

$A = 1\times 1$, $A = [a_{11}]$, $A^T = [a_{11}]$.

$$\det A = \det A^T$$

Suppose $\det A = \det A^T$ for all $K\times K$ matrices.

$A = (K+1)\times(K+1)$.

$$\det A = \sum_{j=1}^{K+1} A(1,j)(-1)^{1+j}\det(A_{1,j})$$

co-factor expansion along row 1.

$$\det A^T = \sum_{j=1}^{K+1} (A^T)(j,1)(-1)^{j+1}\det((A^T)_{j,1})$$

co-factor expansion along column 1.

*) $(-1)^{j+1} = (-1)^{1+j}$

*) $(A^T)(j,1) = A(1,j)$

*) $(A^T)_{j,1}$ = sub matrix obtained by removing the j-th row and 1st Column of A^T

$= \begin{pmatrix}\text{sub matrix obtained by removing} \\ \text{1st row and the } j\text{-th column of } A\end{pmatrix}^T$

*) $(A^T)_{j,1} = \underbrace{(A_{1,j})^T}_{}$ $K \times K$

$$\det\left((A^T)_{j,1}\right) = \det\left((A_{1,j})^T\right)$$
$$= \det(A_{1,j})$$

$\Rightarrow \det A = \det A^T$

Q: $\det(AB) = ?$

Theorem 6 (Chap 3)

Suppose both A and B are $n \times n$. Then we have

$$\boxed{\det(AB) = \det A \cdot \det B}$$

proof: Recall

*). Both A and B are invertible

$\Rightarrow$ $A \cdot B$ is invertible (Theorem 6, chap 2)

*). AB is invertible $\Rightarrow$ Both A and B are invertible.

(homework prob).

Case 1. A is not invertible.

$\Rightarrow \det A = 0$.

A is not invertible

$\Rightarrow$ AB is not invertible.

$\Rightarrow$ $\det(AB) = 0$.

$\Rightarrow$ $\det(AB) = \det(A)\det(B)$ is valid

Case 2: A is invertible.

$\Rightarrow$ A is equivalent to I.

$$A \xrightarrow{\text{ERO's}} I.$$

$$E_p \cdots E_2 E_1 A = I$$

$$\Rightarrow \det A = \underbrace{(-1)^r k_1 \cdots k_q}_{\substack{\text{effects of EROs} \\ \neq 0.}} \det I.$$

$$\det I = 1$$

$$\Rightarrow \det A = (-1)^r k_1 \cdots k_q$$

$$(AB) \xrightarrow{\text{The same sequence of EROs}} B$$

$$E_p \cdots E_2 E_1 (AB) = (E_p \cdots E_2 E_1 A) B$$
$$= I \cdot B = B$$

$$\det(AB) = \underbrace{(-1)^r k_1 \cdots k_q}_{\text{effects of EROs}} \det B$$

$$= \det A \cdot \det B .$$

$\Longrightarrow \det(AB) = \det A \cdot \det B$ is valid

end of proof

In general, $\det(A+B) \neq \det A + \det B$.

Ex. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$

$\det A = 1$, $\det B = 0$

$\det(A+B) = \det \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = 1\times1 - 3\times1 = -2$.

$$\underbrace{\det A}_{1} + \underbrace{\det B}_{0} \neq \underbrace{\det(A+B)}_{-2}$$

properties of determinants

1) $A \xrightarrow{\text{row reduction}} B$ (echelon form).

$$\det A = \underbrace{(-1)^r k_1 \cdots k_q}_{\text{non-zero}} \det B .$$

2). $\det A \neq 0$ if and only if A is invertible.

3) $\det A = \det A^T$

4) $\det(AB) = \det A \cdot \det B$

5) $\det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots \; 0 & a_{nn} \end{bmatrix} = a_{11} \cdots a_{nn}$

Special case: $\det I = 1$.

6). $\det(\alpha A) = \alpha^n \det A$ $\quad A : n \times n$

Special case. $\det(-A) = (-1)^n \det A$

7). $\det(A^k) = (\det A)^k$

8) $\det(A^{-1}) = (\det A)^{-1}$

proof: $A^{-1}A = I$

$\Rightarrow \det(A^{-1}) \det A = 1 . \Rightarrow \det(A^{-1}) = (\det A)^{-1}$

Matri $A = n \times n$.

$$A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$$

let $A_j(\vec{b})$ = matrix obtained by replacing column j of A with $\vec{b}$

$$A_j(\vec{b}) = [\vec{a}_1 \cdots \vec{a}_{j-1} \ \underset{\text{column } j}{\vec{b}} \ \vec{a}_{j+1} \cdots \vec{a}_n]$$

Cramer's rule for solving $A\vec{x} = \vec{b}$:

Suppose A is invertible.

Then the unique solution of $A\vec{x} = \vec{b}$ is

$$x_j = \frac{\det A_j(\vec{b})}{\det A}, \quad j = 1, 2, \cdots n.$$

Sec 5.1 Eigenvalues and eigenvectors.

Def: A: $n \times n$.

If $\vec{x} \neq \vec{0}$ satisfies $A\vec{x} = \lambda \vec{x}$ for some λ.

then λ is called an eigenvalue ~~and~~ of A and $\vec{x}$ is called an eigenvector corresponding to eigenvalue λ.

Q: Given a vector $\vec{u} \neq \vec{0}$

How to find if $\vec{u}$ is an eigenvector?

Method: Check if $A\vec{u}$ is a multiple of $\vec{u}$.

Ex

$$A = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Find out if $\vec{u}$ and $\vec{v}$ are eigenvectors.

$$A\vec{u} = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2\times(-5)+5\times 6 \\ 6\times(-5)+1\times 6 \end{bmatrix} = \begin{bmatrix} 20 \\ -24 \end{bmatrix} = -4\vec{u}$$

$\Longrightarrow \vec{u}$ is an eigenvector.

$$A\vec{v} = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\times 2+5\times 3 \\ 6\times 2+1\times 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 15 \end{bmatrix}$$

not a multiple of $\vec{v}$

$\Longrightarrow \vec{v}$ is not an eigenvector.

<u>Q</u>: Given a scalar λ.
How to find if λ is an eigenvalue?

λ is an eigenvalue.

$\iff$ $A\vec{x} = \lambda\vec{x}$ has a non-trivial solution.

$\iff$ $A\vec{x} - \lambda I\vec{x} = \vec{0}$ has a non-trivial

$\iff$ $(A - \lambda I)\vec{x} = \vec{0}$ has a non-trivial

$\iff$ $(A - \lambda I)$ is not invertible

$\iff$ $\det(A - \lambda I) = 0$.

<u>Method</u>: Check if $\det(A - \lambda I) = 0$.

<u>Statement</u>: λ is an eigenvalue of A
if and only if $\det(A - \lambda I) = 0$

Ex. $A = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$.

$\lambda_1 = -4$

$\lambda_2 = 3$

Find out if λ_1 and λ_2 are eigenvalues.

$\lambda_1 = -4$

$\det(A - (-4)I) = \det \begin{bmatrix} 2+4 & 5 \\ 6 & 1+4 \end{bmatrix} = 6\times 5 - 6\times 5 = 0$ ✓

$\det(A + 4I)$

$\Rightarrow \lambda_1 = -4$ is an eigenvalue

$\lambda_2 = 3$

$\det(A - 3I) = \det \begin{bmatrix} 2-3 & 5 \\ 6 & 1-3 \end{bmatrix} = (-1)\times(-2) - 6\times 5$

$= -28 \neq 0$ ✗

$\Rightarrow \lambda_2 = 3$ is not an eigenvalue.