02/20/2018

Recap

*) Unique representation of a vector by a basis

Theorem 7, sec. 4.4

*) Coordinate vector of a vector relative to a basis

$B = \{\vec{b}_1, \cdots, \vec{b}_p\}$ a basis of H

$\vec{x}$ in H $\longrightarrow$ $\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p$ (representation is unique)

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \text{ in } \mathbb{R}^p$$

*) One-to-one correspondence between H and $\mathbb{R}^p$

$$\vec{x} \longleftrightarrow [\vec{x}]_B$$

$$H \longleftrightarrow \mathbb{R}^p$$

*) Dimension of a subspace (# of vectors in a basis)

Statement: Two bases of H must have the same # of vectors.

Statement: Matrices $B = n \times q$, $A = n \times p$

Columns of B are linear combinations of columns of A

$\Longleftrightarrow$ There exists a $p \times q$ matrix C such that

$$\underset{n\times q}{B} = \underset{n\times p}{A} \cdot \underset{p\times q}{C}$$

*) rank A = dim Col A

*) The rank matrix theorem (Theorem 14, Chap 2)

$A = m \times n$

1) $\text{rank } A + \dim \text{Nul } A = n$

# of columns in A

# of variables in $A\vec{x} = \vec{0}$

# of pivot positions

# of basic variables

# of free variables

2) $\text{rank } A = \text{rank } A^T$

(We will prove this today)

*) The invertible matrix theorem (Theorem 8, Chap 2)

$\boxed{A = n \times n}$

a) A is invertible

⋮

m) The columns of A form a basis for $\mathbb{R}^n$

⋮

p) $\text{rank } A = n$

⋮

r) $\dim \text{Nul } A = 0$

*) The spanning set theorem (Theorem 5, Sec 4.3)

*) The basis theorem (Theorem 15, Chap)

So far we studied <u>column vectors</u>.

<u>Def</u>: A row vector = a $1 \times n$ matrix

$$\vec{u} = [u_1 \ u_2 \cdots u_n]$$

$A = m \times n$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \vec{a_1} \\ \vdots \\ \vec{a_m} \end{bmatrix}$$

$\vec{a_i} = [a_{i1} \cdots a_{in}]$

$\vec{a_i}$ is in $\mathbb{R}^n$

<u>Def</u>. Row space of A

$$\text{Row} A \equiv \text{span}\{\vec{a_1}, \vec{a_2}, \cdots, \vec{a_m}\}$$

We study <u>a basis for Row A</u> and <u>dim Row A</u>

We do row reduction on A to reduced echelon form B

$$A \xrightarrow{\text{row reduction}} B \ (\underline{REF}).$$

<u>Statement 1</u>:

Non-zero rows <u>of B</u> form a basis for <u>Row B</u>

<u>Statement 2</u>

$$\text{Row } A = \text{Row } B$$

Statement 3:

Non-zero rows of B form a basis for Row A

Statement 4:

# of pivot positions in A

$= \dim \text{Row } A$

$= \dim \text{Col } A^T$

$= \text{rank } A^T$

$$\Rightarrow \boxed{\text{rank } A = \text{rank } A^T}$$

Sec 3.1. The determinant of a matrix.

Def: $A : n \times n$

$n = 1$. $A = [a_{11}]$

$\underbrace{\det A}_{\substack{\text{notation for} \\ \text{determinant of } A}} = a_{11}$

Suppose $\det A$ is defined for $k \times k$ matrices.

For $n = k+1$,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

let A_{ij} be the sub-matrix obtained by removing the i-th row and the j-th column of A

A_{ij} : $(n-1)\times(n-1)$, $K \times K$.

$$\det A = \sum_{j=1}^{n} a_{1j} \underline{(-1)^{1+j} \det A_{1j}}$$

Caution: <u>A must be square</u>.

Ex. A : 2×2. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det A = a_{11} \underbrace{(-1)^{1+1}}_{=1} \underbrace{\det A_{11}}_{a_{22}} + a_{12} \underbrace{(-1)^{1+2}}_{=-1} \underbrace{\det A_{12}}_{a_{21}}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

$$\boxed{\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}}$$

Def: (co-factor)

The (i,j) co-factor of A is defined as

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}$$

Co-factor expansion along row 1

Theorem 1:

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$

co-factor expansion along row i

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij}$$

co-factor expansion along column j

Proof: We show

$$\det A = \sum_{\substack{(j_1, j_2, \cdots, j_n) \text{ is} \\ \text{a permutation of} \\ (1,2,3,\cdots,n) \\ \text{Sum over all} \\ \text{permutations}}} \underline{\varepsilon(j_1, j_2, \cdots, j_n)}\, a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n}.$$

$$\underset{\text{Levi-Civita Symbol}}{\varepsilon(j_1, j_2, \cdots, j_n)} = \begin{cases} +1, & \text{if it is an even permutation} \\ -1, & \text{if it is an odd permutation} \end{cases}$$

Ex. $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & 1 & 1 \end{bmatrix}$

$$\det A = a_{11}(-1)^{1+1}\det A_{11} + a_{12}(-1)^{1+2}\det A_{12} + a_{13}(-1)^{1+3}\det A_{13}$$

$$= 1\det\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} + (-1)(-1)\det\begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$+ 1\det\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= 2-(-2) + [-1-(-4)] + [-1-4]$$

$$= 4+3-5 = 2$$

Ex. $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 2 & -2 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$$\underline{\det A} = a_{44}(-1)^{4+4}\det A_{44} = 2\times\det\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$= 2\times a_{32}(-1)^{3+2}\det A_{32}$$

$$= 2\times 2\times(-1)\det\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = -4[-2-(-1)] = 4$$

Theorem 2.

For a triangular matrix

$$A = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & 0 & \ddots & \vdots \\ 0 & & 0 & a_{nn} \end{bmatrix}$$

$$\det A = a_{11} a_{22} \cdots a_{nn}$$

proof: we expand along column 1

$$\det A = a_{11}(-1)^{1+1} \det A_{11}$$

$$= a_{11} \det A_{11} = \cdots$$

$$= a_{11} a_{22} \cdots a_{nn}$$

Ex. $A = \begin{bmatrix} 2 & 7 & 5 & 9 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

$$\det A = 2 \times 1 \times 3 \times 5 = 30$$

Co-factor expansion is computationally expensive.

let Cost(n) = # of operations required in calculating det of an $n \times n$ matrix.

$\text{Cost}(2) = 2$

$$\det A = \sum_{j=1}^{n} a_{1j} (-1)^{1+j} \det A_{1j}$$

$\text{Cost}(n) \geq n \times \text{cost}(n-1)$

$\Rightarrow \quad \text{cost}(n) \geq n!$

$\text{Cost}(25) \geq 25! = 1.5 \times 10^{25}$

Sec 3.2 Properties of determinants

Theorem 3 (Effects of EROs on the determinant)

a). $A \xrightarrow{\text{Add } c \times (\text{row } k) \text{ to } (\text{row } l)} B$

$\det B = \det A$

b) $A \xrightarrow{\text{Interchange } (\text{row } k) \text{ and } (\text{row } l)} B$

$\det B = -\det A$

c). $A \xrightarrow{\text{Multiply } (\text{row } k) \text{ by } r \neq 0} B$

$\det B = r \det A$

proof c) Expand along row k.

$$\det B = \sum_{j=1}^{n} r a_{kj} (-1)^{k+j} \det A_{kj}$$

$$= r \underbrace{\sum_{j=1}^{n} a_{kj} (-1)^{k+j} \det A_{kj}} = r \det A$$

a), b) . Mathematical induction.

$n=2$. a). $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow[\text{to row 2}]{\text{Add } c \times \text{row 1}} \begin{bmatrix} a_{11} & a_{12} \\ a_{21}+ca_{11} & a_{22}+ca_{12} \end{bmatrix}$

$\quad\quad A \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad B$

$$\det B = a_{11}(a_{22}+ca_{12}) - (a_{21}+ca_{11})a_{12}$$
$$= a_{11}a_{22} - a_{21}a_{12} = \det A.$$

b). $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{\text{Interchange}} \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$

$\quad\quad A \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad B.$

$$\det B = a_{21}a_{12} - a_{11}a_{22} = -\det A$$

How to use this theorem

a). $\det A = \det B$.

b). $\det A = -\det B$

c) $\det A = \frac{1}{r}\det B$

procedure of calculating det A using EROs

*) Row reduce A to B_1, $\rightarrow B_2$, $\rightarrow B_3$, $\cdots \rightarrow B_N$

Echelon form

*) Include the effect of each ERO we used

$$A \xrightarrow[\cancel{\det A}]{\text{Add } c \times (\text{row } k) \text{ to } (\text{row } l)} B.$$

$$\det A = \det B$$

$$A \xrightarrow{\text{Interchange two rows}} B$$

$$\det A = -\det B$$

$$A \xrightarrow{\text{multiply } (\text{row } k) \text{ by } r \neq 0} B$$

$$\det A = \frac{1}{r} \det B$$

Ex. $A = \begin{bmatrix} 3 & -9 & 6 \\ -2 & 6 & 1 \\ 1 & -4 & 5 \end{bmatrix}$ Multiply (row 1) by $1/3$.

$$\det A = 3 \det \begin{bmatrix} 1 & -3 & 2 \\ -2 & 6 & 1 \\ 1 & -4 & 5 \end{bmatrix}$$ Add $2 \times$ (row 1) to (row 2)
Add $-1 \times$ (row 1) to (row 3)

$$= 3 \det \begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 5 \\ 0 & -1 & 3 \end{bmatrix}$$ Interchange (row 2) and (row 3)

$$= 3(-1) \det \begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$ ← This is already in echelon form

$$= 3(-1)\, 1\, (-1)\, 5 = 15$$

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \\ 5 & -5 & 6 \end{bmatrix}$ Add $2 \times R1$ to $R2$
Add $(-5) \times R1$ to $R3$

$$\det A = \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 10 \\ 0 & -15 & -9 \end{bmatrix}$$ Multiply $R2$ by $1/5$

$$= 5 \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -15 & -9 \end{bmatrix}$$ Add $15 \times R2$ to $R3$

$$= 5 \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 21 \end{bmatrix} \leftarrow$$ This is already in echelon form.

$$= 5 \cdot 1 \cdot 1 \cdot 21 = 105$$