AMS 10/10A, Homework 7

Problems for Section 2.8 and 2.9

Problem 1. Suppose A is $m \times n$. Prove the following equality

$$dim\ Col(A) + dim\ Nul(A^T) \quad = \quad m$$

Problem 2. Suppose A is $m \times n$ and b is in R^m. Prove that if the equation $Ax = b$ is consistent, then $rank\ [A, b] = rank\ A$.

Problem 3. Suppose A is 5×8 and $rank\ A = 5$.
Does $Ax = 0$ have a non-trivial solution? Why?
Does $A^T x = 0$ have a non-trivial solution? Why?

Problems for Section 3.1 and 3.2

Problem 4. Compute the determinant of each of the following matrices.
You can use either co-factor expansion or row reduction.

$$A = \begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 4 \\ 5 & -7 & 6 \end{bmatrix},$$

$$C = \begin{bmatrix} 6 & 2 & 5 & 4 \\ 7 & 3 & 6 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 0 & 6 & 4 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Problem 5. Show that for arbitrary real numbers a, b, c, and d, the determinant of the following matrix is always zero.

$$\begin{bmatrix} a & 0 & d & c \\ b & 0 & -c & d \\ 0 & c & -b & a \\ 0 & d & a & b \end{bmatrix}$$

Problem 6. Find the value(s) of a for which the determinant of the following matrix is zero.

$$\begin{bmatrix} a & \sqrt{2} & 0 \\ \sqrt{2} & a & \sqrt{2} \\ 0 & \sqrt{2} & a \end{bmatrix}$$

Problem 7. Let A and B be 4×4 square matrices such that $det(A) = 3$ and $det(B) = -2$. Compute $det(2A)$, $det(A^3)$, $det(A^{-1})$, $det(A^2B^3)$ and $det(A^3B^{-2})$.

Problem 8. Prove that $det(AA^T)$ is nonnegative for any $n \times n$ matrix A.

Problem 9. Let A be an $n \times n$ matrix and let P be an $n \times n$ invertible matrix. Prove that $det(P^{-1}AP) = det(A)$.

Problem 10. Let A be an $n \times n$ matrix such that $A^T = -A$. Prove that A is not invertible if n is odd.

Problem 11. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Show that i) $A^T = -A$ and ii) A is invertible.
Does this result contradict the conclusion in Problem 10 above?

Problem 12. Suppose $det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5$. Find

$$det \begin{bmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{bmatrix},$$

$$det \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix},$$

$$det\left(3 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right),$$

$$det \begin{bmatrix} a & b & c \\ d & e & f \\ a & b & c \end{bmatrix}$$