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Recap

Subspace, definition of a subspace

3 conditions

OR 1 unified condition

Col A: A: $m\times n$ matrix

$A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$

$\text{Col } A \equiv \text{span}\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$

Nul A: $\text{Nul } A \equiv$ solution set of $A\vec{x} = \vec{0}$

A basis for a subspace H, 2 conditions

Definition: *) $\{\vec{a}_1, \cdots, \vec{a}_p\}$ is linearly independent

*) $\text{Span}\{\vec{a}_1, \cdots, \vec{a}_p\} = H$

A basis for Col A

*) Use echelon form to identify pivot columns

*) Take pivot columns of matrix A.

*) DO NOT take pivot columns of echelon form.

$\boxed{\text{Theorem 13}}$: The pivot columns of A form a basis for Col A

A very simple example.

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}$ is a basis for Col A

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is <u>NOT</u> a basis for Col A

<u>A basis for Nul A</u>

{ vectors in the solution set in parametric form }

<u>Q</u>: For a given $\vec{b}$, is $\vec{b}$ in Col A?
How to check?

<u>Q</u>: Is Col $A = \mathbb{R}^m$? $\quad A$: $m \times n$ matrix

<u>Q</u>: For a given $\vec{u}$, is $\vec{u}$ in Nul A?
How to check?

Sec 2.9 Dimension and rank

dimension of a subspace

rank of a matrix

Theorem 7. of sec 4.4 (Unique representation theorem)

Let $B = \{\vec{b_1}, \vec{b_2}, \cdots \vec{b_p}\}$ be a basis for a subspace H.

Then, for each $\vec{x}$ in H, there is one and only one set of coefficients $\{c_1, c_2, \cdots, c_p\}$ such that

$$\vec{x} = c_1 \vec{b_1} + c_2 \vec{b_2} + \cdots + c_p \vec{b_p}$$

proof: "Existence"

$$\text{Span}\{\vec{b_1}, \vec{b_2}, \cdots, \vec{b_p}\} = H$$

$$\Longrightarrow \quad \vec{x} = c_1 \vec{b_1} + c_2 \vec{b_2} + \cdots + c_p \vec{b_p}$$

"Uniqueness"

Suppose we also have

$$\vec{x} = d_1 \vec{b_1} + d_2 \vec{b_2} + \cdots + d_p \vec{b_p}$$

$$\Longrightarrow \quad \vec{0} = (c_1 - d_1)\vec{b_1} + (c_2 - d_2)\vec{b_2} + \cdots + (c_p - d_p)\vec{b_p}$$

$\{\vec{b_1}, \cdots, \vec{b_p}\}$ is linearly independent

$$\Longrightarrow c_1 = d_1, \ c_2 = d_2, \ \cdots, \ c_p = d_p$$

Definition (coordinate of a vector relative to a basis)

Let $B = \{\vec{b}_1, \cdots, \vec{b}_p\}$ be a basis for a subspace H.

Each $\vec{x}$ in H is uniquely represented by

$$\vec{x} = c_1 \vec{b}_1 + \cdots + c_p \vec{b}_p$$

The vector

$$[\vec{x}]_B \equiv \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

is called the coordinate vector of vector $\vec{x}$ relative to basis B.

*). $[\vec{x}]_B$ is in $\mathbb{R}^p$

*). $\vec{x} \xleftrightarrow[\text{Theorem}]{\text{Unique representation}} [\vec{x}]_B$

*) $H \xleftrightarrow[\text{one to one correspondence}]{} \mathbb{R}^p$.

Ex. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$

Let $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$

$B = \{\vec{v}_1, \vec{v}_2\}$ is a basis for H.

Q: Is $\vec{x}$ in H?

If so, find $[\vec{x}]_B$

We consider the vector equation.

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x}$$

$$c_1\begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

The augmented matrix

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ -4 & 6 & 3 \\ -3 & 7 & -4 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & -9/2 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{array}\right]$$

The right-most column is not a pivot column.

$\Rightarrow$ It is consistent.

A solution. $\begin{cases} c_1 = -9/2 \\ c_2 = -5/2 \end{cases}$

$$[\vec{x}]_B = \begin{bmatrix} -9/2 \\ -5/2 \end{bmatrix},$$

$[\vec{x}]_B$ is in $\mathbb{R}^2$

$\vec{v}_1, \vec{v}_2, \vec{x}$ are in $\mathbb{R}^3$

$$H \xleftrightarrow[\text{correspondence}]{\text{one-to-one}} \mathbb{R}^2$$

Definition (dimension of a subspace)

Special case $\quad \dim\{\vec{0}\} = 0$

For $H \neq \{\vec{0}\}$

$$\dim H \equiv \text{the number of vectors in a basis for } H$$

$\boxed{\text{Statement}}$: let H be a subspace of $\mathbb{R}^n$

Two bases for H must have the same number of vectors.

proof: Suppose H has two bases

$\{\vec{a}_1, \cdots, \vec{a}_p\}$

$\{\vec{b}_1, \cdots, \vec{b}_q\}$

suppose $q > p$:

let $A = [\vec{a}_1 \cdots \vec{a}_p] \quad n \times p$.

$B = [\vec{b}_1 \cdots \vec{b}_q] \quad n \times q$

$\boxed{\vec{b}_j = A\,\vec{c}_j} \qquad \vec{c}_j$ is in $\mathbb{R}^p$.

let $C = [\vec{c}_1 \cdots \vec{c}_q] \ : \ p \times q$

$$\Rightarrow \underset{n\times q}{B} = \underset{n\times p}{A}\ \underset{p\times q}{C}$$

$C: \ p \times q \qquad \underline{q > p}$

$\Rightarrow$ Columns of C are linearly dependent.

$C\vec{x} = \vec{0}$ has a non-trivial solution.

$C\vec{d} = \vec{0}, \quad \vec{d} \neq \vec{0}$

$B\vec{d} = (AC)\vec{d} = A(C\vec{d}) = A\vec{0} = \vec{0}$

$\Rightarrow$ $B\vec{x} = \vec{0}$ has a non-trivial solution.

$\Rightarrow$ Columns of B are linearly dependent

$\Rightarrow$ $\boxed{\text{Contradiction}}$

$\Rightarrow$ $q > p$ is impossible.

end of proof

Ex: $\quad \vec{u} \neq \vec{0}$

$\dim \text{Span}\{\vec{u}\} = 1$

$\dim \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}3\\1\\2\end{bmatrix}\right\} = 2.$

Definition. (rank of a matrix)

$\text{rank}\, A \equiv \dim \text{Col}\, A$

$$\boxed{\begin{aligned}\dim \operatorname{Col} A &= \#\text{ of pivot columns} \\ &= \#\text{ of pivot positions}\end{aligned}}$$

Statement rank A = # of pivot columns in A
= # of pivot positions in A

Ex
$$A = \begin{bmatrix} 1 & 3 & -1 & -3 \\ 2 & 6 & 1 & 9 \\ 1 & 3 & -2 & -8 \end{bmatrix}$$

Find rank A

$$\xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & 0 & 3 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ rank $A = 2$

Statement A : $m \times n$

$\dim \operatorname{Nul} A$ = # of free variables
= (# of variables) − (# of basic variables)
= n − (# of pivot positions in A)

$\boxed{\#\text{ of columns in } A}$

Theorem 14 (The rank theorem)

A: $m \times n$. We have

*). $\operatorname{rank} A^T = \operatorname{rank} A$ (we will prove this later)

*) $\operatorname{rank} A + \dim \operatorname{Nul} A = n$ ✓

Revisit Theorem 8 (invertible matrix theorem)

A: $n \times n$

All statements below are equivalent.

a) A is invertible

...

c) A has n pivot positions

d) $A\vec{x} = \vec{0}$ has only the trivial solution

e) The columns of A are linearly independent.

⋮

h) The columns of A span $\mathbb{R}^n$

⋮

l) A^T is invertible.

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m). The columns of A form a basis for $\mathbb{R}^n$

m) $\Longleftrightarrow$ e) + h)

n). $\operatorname{Col} A = \mathbb{R}^n$

n) $\Longleftrightarrow$ h)

o) $\dim \operatorname{Col} A = n$

o) $\Longleftrightarrow$ c)

p) $\operatorname{rank} A = n$

p) $\Longleftrightarrow$ o)

q) $\operatorname{Nul} A = \{\vec{0}\}$

q) $\Longleftrightarrow$ d)

r) $\dim \operatorname{Nul} A = 0$.

r) $\Longleftrightarrow$ q)

Theorem 15 (The basis theorem)

Suppose H is a subspace in $\mathbb{R}^n$ and $\dim H = p > 0$

1) If $\{\vec{b}_1, \cdots \vec{b}_p\}$ in H is linearly independent, then $\{\vec{b}, \cdots, \vec{b}_p\}$ is a basis for H.

2) If $\operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_p\} = H$, then $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is a basis for H.

proof: 1) We need to show $\operatorname{span}\{\vec{b}_1, \cdots, \vec{b}_p\} = H$

Let $\{\vec{a}_1, \cdots, \vec{a}_p\}$ be a basis for H.

Let $A = [\vec{a}_1 \cdots \vec{a}_p]$ $\quad n \times p$

$B = [\vec{b}_1 \cdots \vec{b}_p]$ $\quad n \times p$

$\Rightarrow$ There exists matrix C: $(p \times p)$ such that

$$\boxed{\underset{n\times p}{B} = \underset{n\times p}{A}\,\underset{p\times p}{C}}$$

$\{\vec{b}_1, \cdots, \vec{b}_p\}$ is linearly independent

$\Rightarrow$ $B\vec{x} = \vec{0}$ has only the trivial solution.

$$B\vec{x} = (A\,C)\vec{x} = A(C\vec{x})$$

$\Rightarrow$ $C\vec{x} = \vec{0}$ has only the trivial solution.

$\Rightarrow$ C is invertible.

$\Rightarrow$ $BC^{-1} = A$

$\Rightarrow$ $\boxed{A = BC^{-1}}$

Columns of A are linear combinations of columns of B.

$\Rightarrow$ $\text{span}\{\vec{b}_1, \cdots, \vec{b}_p\} = H$.

2). We need to show $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is linearly independent.

Theorem 5 of sec 4.3 (The spanning set theorem)

Consider $H = \text{span}\{\vec{v}_1, \cdots \vec{v}_p\} \neq \{\vec{0}\}$

a). If $\vec{v}_p$ is a linear combination of others, then

$$\text{span}\{\vec{v}_1, \cdots, \vec{v}_{p-1}\} = H$$

b). $\{\vec{v}_1, \cdots, \vec{v}_p\}$ or a smaller set of $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is a basis for H.

Suppose $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is linearly dependent.

$\Rightarrow$ A smaller set of $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is a basis.

$\Rightarrow$ $\dim H < p$

$\Rightarrow$ Contradicting with $\underline{\dim H = p}$

$\Rightarrow$ $\{\vec{v}_1, \cdots, \vec{v}_p\}$ must be linearly independent!