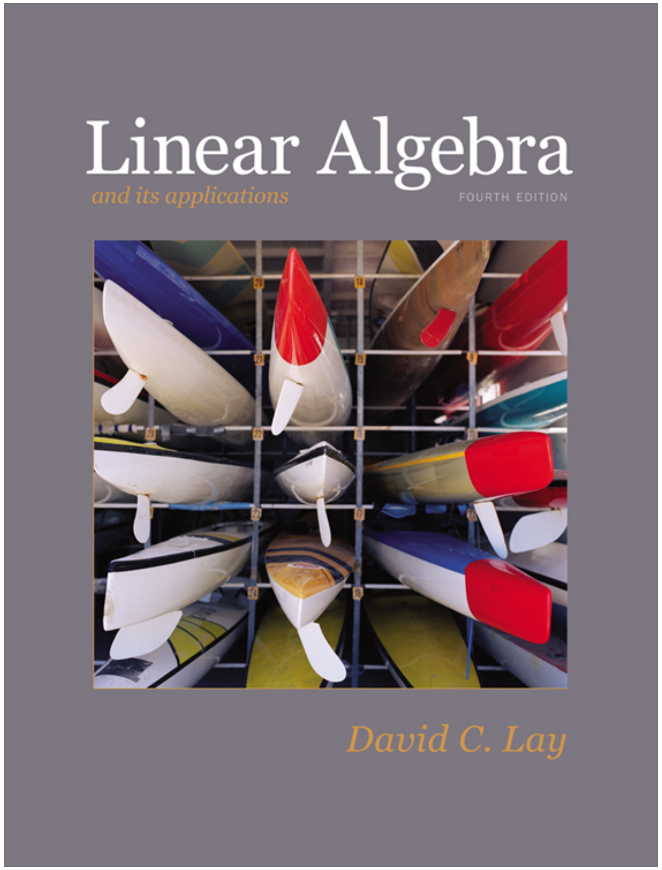
2 Matrix Algebra

2.2

THE INVERSE OF A MATRIX



PEARSON

MATRIX OPERATIONS

- An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

where $I = I_n$, the $n \times n$ identity matrix.

- In this case, C is an inverse of A.
- In fact, C is uniquely determined by A, because if B were another inverse of A, then

$$B = BI = B(AC) = (BA)C = IC = C.$$

- This unique inverse is denoted by A^{-1}, so that

$$A^{-1}A = I \text{ and } AA^{-1} = I.$$

MATRIX OPERATIONS

- **Theorem 5:** If A is an invertible $n \times n$ matrix, then for each **b** in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
- **Proof:** Take any **b** in $\mathbb{R}^n$.
- A solution exists because if $A^{-1}\mathbf{b}$ is substituted for **x**, then $A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}$.
- So $A^{-1}\mathbf{b}$ is a solution.
- To prove that the solution is unique, show that if **u** is any solution, then **u** must be $A^{-1}\mathbf{b}$.
- If $A\mathbf{u} = \mathbf{b}$, we can multiply both sides by A^{-1} and obtain $A^{-1}A\mathbf{u} = A^{-1}\mathbf{b}$, $I\mathbf{u} = A^{-1}\mathbf{b}$, and $\mathbf{u} = A^{-1}\mathbf{b}$.

MATRIX OPERATIONS

- **Theorem 6:**
  a. If A is an invertible matrix, then A^{-1} is invertible and
  $$(A^{-1})^{-1} = A$$
  b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is, $$(AB)^{-1} = B^{-1}A^{-1}$$
  c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is, $$(A^T)^{-1} = (A^{-1})^T$$

MATRIX OPERATIONS

- **Proof:** To verify statement (a), find a matrix C such that

$$A^{-1}C = I \text{ and } CA^{-1} = I$$

- These equations are satisfied with A in place of C. Hence A^{-1} is invertible, and A is its inverse.
- Next, to prove statement (b), compute:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

- A similar calculation shows that $(B^{-1}A^{-1})(AB) = I$.
- For statement (c), use Theorem 3(d), read from right to left, $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$.
- Similarly, $A^T(A^{-1})^T = I^T = I$.

ELEMENTARY MATRICES

- Hence A^T is invertible, and its inverse is $(A^{-1})T$.
- The generalization of Theorem 6(b) is as follows:

  The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.
- An invertible matrix A is row equivalent to an identity matrix, and we can find A^{-1} by watching the row reduction of A to I.
- An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

ELEMENTARY MATRICES

- **Example 1:** Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

  $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

  Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A.

ELEMENTARY MATRICES

- **Solution:** Verify that

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}, \; E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$

$$E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

- Addition of -4 times row 1 of A to row 3 produces E_1A.

ELEMENTARY MATRICES

- An interchange of rows 1 and 2 of A produces E_2A, and multiplication of row 3 of A by 5 produces E_3A.

- Left-multiplication by E_1 in Example 1 has the same effect on any $3 \times n$ matrix.

- Since $E_1 \cdot I = E_1$, we see that E_1 itself is produced by this same row operation on the identity.

ELEMENTARY MATRICES

- Example 1 illustrates the following general fact about elementary matrices.

- If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

- Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

ELEMENTARY MATRICES

- **Theorem 7:** An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.
- **Proof:** Suppose that A is invertible.
- Then, since the equation $A\mathrm{x} = \mathrm{b}$ has a solution for each **b** (Theorem 5), A has a pivot position in every row.
- Because A is square, the n pivot positions must be on the diagonal, which implies that the reduced echelon form of A is I_n. That is, $A \sim I_n$.

ELEMENTARY MATRICES

- Now suppose, conversely, that $A \sim I_n$.
- Then, since each step of the row reduction of A corresponds to left-multiplication by an elementary matrix, there exist elementary matrices $E_1, \ldots, E_p$ such that

$$A \sim E_1 A \sim E_2(E_1 A) \sim \ldots \sim E_p(E_{p-1} \ldots E_1 A) = I_n.$$

- That is, $E_p \ldots E_1 A = I_n$ ----(1)
- Since the product $E_p \ldots E_1$ of invertible matrices is invertible, (1) leads to

$$(E_p \ldots E_1)^{-1}(E_p \ldots E_1)A = (E_p \ldots E_1)^{-1} I_n$$

$$A = (E_p \ldots E_1)^{-1}.$$

ALGORITHM FOR FINDING A^{-1}

- Thus A is invertible, as it is the inverse of an invertible matrix (Theorem 6). Also,

$$A^{-1} = \left[(E_p \dots E_1)^{-1} \right]^{-1} = E_p \dots E_1.$$

- Then $A^{-1} = E_p \dots E_1 \cdot I_n$, which says that A^{-1} results from applying $E_1, \dots, E_p$ successively to I_n.
- This is the same sequence in (1) that reduced A to I_n.
- Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A does not have an inverse.

ALGORITHM FOR FINDING A^{-1}

- **Example 2:** Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}, \text{ if it exists.}$$

- **Solution:**

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

ALGORITHM FOR FINDING A^{-1}

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

ALGORITHM FOR FINDING A^{-1}

- Theorem 7 shows, since $A \sim I$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}.$$

- Now, check the final answer.

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ANOTHER VIEW OF MATRIX INVERSION

- It is not necessary to check that $A^{-1}A = I$ since A is invertible.

- Denote the columns of I_n by $\mathbf{e}_1,\ldots,\mathbf{e}_n$.

- Then row reduction of $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$ can be viewed as the simultaneous solution of the n systems

$$A\mathrm{x} = \mathrm{e}_1,\ A\mathrm{x} = \mathrm{e}_2,\ \ldots,\ A\mathrm{x} = \mathrm{e}_n \qquad \text{----(2)}$$

where the "augmented columns" of these systems have all been placed next to A to form

$$\begin{bmatrix} A & \mathrm{e}_1 & \mathrm{e}_2 & \cdots & \mathrm{e}_n \end{bmatrix} = \begin{bmatrix} A & I \end{bmatrix}.$$

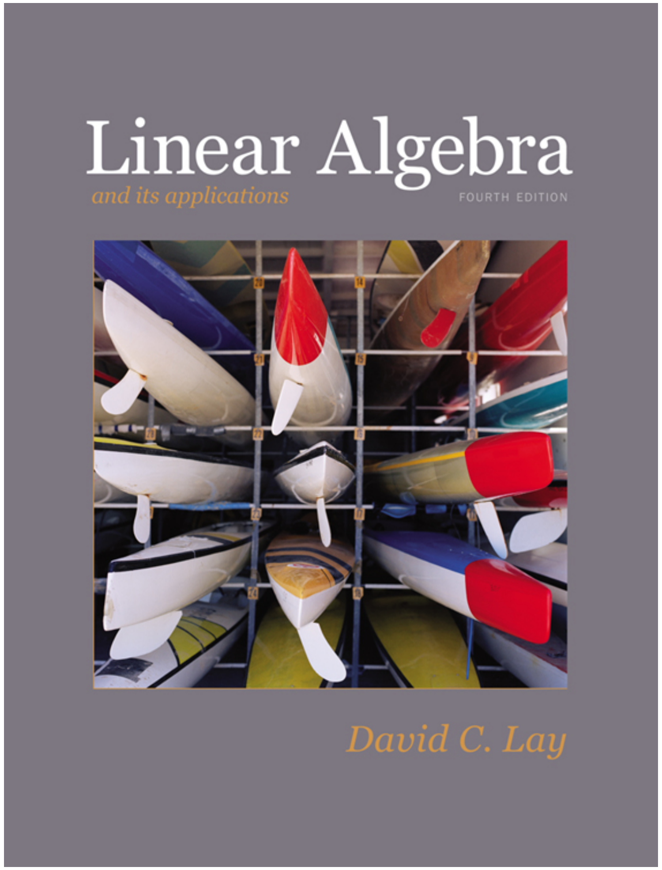
ANOTHER VIEW OF MATRIX INVERSION

- The equation $AA^{-1} = I$ and the definition of matrix multiplication show that the columns of A^{-1} are precisely the solutions of the systems in (2).

2 Matrix Algebra

2.3

CHARACTERIZATIONS OF INVERTIBLE MATRICES



THE INVERTIBLE MATRIX THEOREM

- **Theorem 8:** Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

    a. A is an invertible matrix.

    b. A is row equivalent to the $n \times n$ identity matrix.

    c. A has n pivot positions.

    d. The equation $A\mathbf{x} = 0$ has only the trivial solution.

    e. The columns of A form a linearly independent set.

THE INVERTIBLE MATRIX THEOREM

f. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

g. The columns of A span $\mathbb{R}^n$.

h. A^T is an invertible matrix.

THE INVERTIBLE MATRIX THEOREM

- Theorem 8 could also be written as "The equation $A\mathrm{x} = \mathrm{b}$ has a unique solution for each **b** in $\mathbb{R}^n$."
- This statement implies (b) and hence implies that A is invertible.
- The following fact follows from Theorem 8.

  Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.
- The Invertible Matrix Theorem divides the set of all $n \times n$ matrices into two disjoint classes: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices.

THE INVERTIBLE MATRIX THEOREM

- Each statement in the theorem describes a property of every $n \times n$ invertible matrix.

- The *negation* of a statement in the theorem describes a property of every $n \times n$ singular matrix.

- For instance, an $n \times n$ singular matrix is *not* row equivalent to I_n, does *not* have n pivot position, and has linearly *dependent* columns.

THE INVERTIBLE MATRIX THEOREM

- **Example 1:** Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

- **Solution:**

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

THE INVERTIBLE MATRIX THEOREM

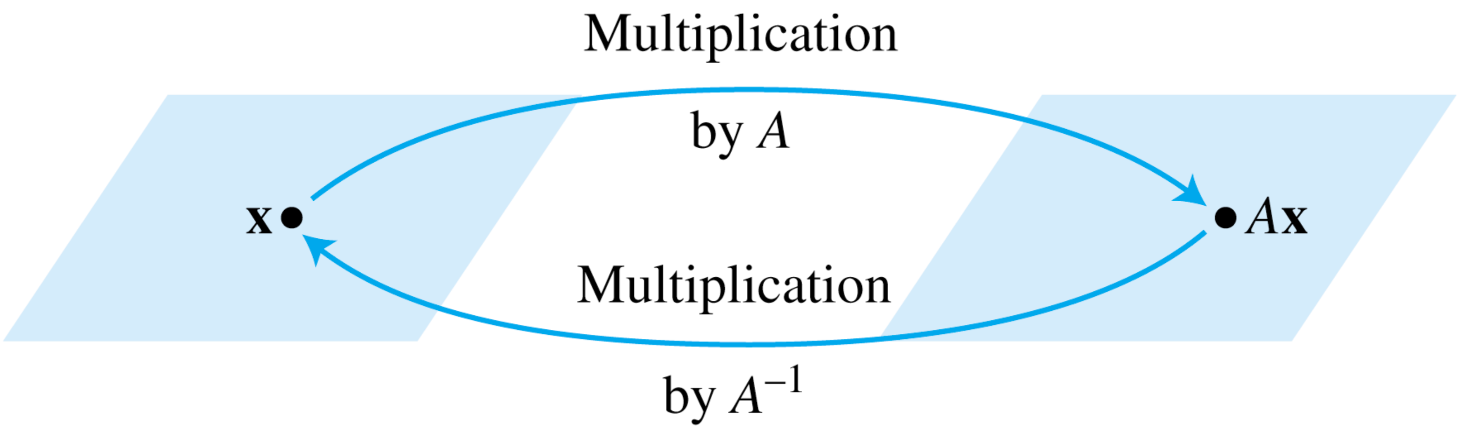
- So A has three pivot positions and hence is invertible, by the Invertible Matrix Theorem, statement (c).

- The Invertible Matrix Theorem *applies only to square matrices*.

- For example, if the columns of a 4×3 matrix are linearly independent, we cannot use the Invertible Matrix Theorem to conclude anything about the existence or nonexistence of solutions of equation of the form $A\mathrm{x} = \mathrm{b}$.

INVERTIBLE LINEAR TRANSFORMATIONS

- Matrix multiplication corresponds to composition of linear transformations.
- When a matrix A is invertible, the equation $A^{-1}A\mathrm{x} = \mathrm{x}$ can be viewed as a statement about linear transformations. See the following figure.



A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.