02/08/2018

$\boxed{\text{Recap}}$ The algorithm for finding A^{-1}

$$[A \mid I] \xrightarrow[\text{reduction}]{\text{row}} [I \mid \square]$$

$$A^{-1} = \square$$

Theorem 8 (Chap 2) Invertible matrix theorem

Matrix A is $n \times n$, $A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$

All statements below are equivalent to each other

a) A is invertible

...

c) # of pivot positions $= n$

d) $A\vec{x} = \vec{0}$ has only the trivial solution

e) $\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$ is linearly independent.

...

h) $\text{Span}\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\} = \mathbb{R}^n$

j) There is an $n \times n$ matrix C such that

$$CA = I$$

k) There is an $n \times n$ matrix D such that

$$AD = I$$

Theorem 8 Supplemental

Let A and B be $n \times n$ matrices

If $AB = I$, then both A and B are invertible,

and $A^{-1} = B$

$B^{-1} = A$

To verify A is invertible, we only need to find C such that $CA = I$ or $AC = I$.

Sec. 2.8 Subspace of $\mathbb{R}^n$

Definition: A subspace is a set H in $\mathbb{R}^n$ that satisfies.

a) $\vec{0}$ is in H

b) If $\vec{u}$ and $\vec{v}$ are in H, then $\vec{u}+\vec{v}$ is in H.

c) If $\vec{u}$ is in H, then $c\vec{u}$ is in H.

Conditions a), b) and c) can be combined into one unified condition.

$$\boxed{\text{If } \vec{u} \text{ and } \vec{v} \text{ are in } H, \text{ then } c_1\vec{u}+c_2\vec{v} \text{ is in } H.}$$

Ex. $\vec{v}_j$ is in $\mathbb{R}^n$

$H = \text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\}$ is a subspace of $\mathbb{R}^n$.

Verify a). $\vec{0} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_p$ ✓

b) $\vec{u} = s_1 \vec{v}_1 + s_2 \vec{v}_2 + \cdots + s_p \vec{v}_p$

$\vec{w} = t_1 \vec{v}_1 + t_2 \vec{v}_2 + \cdots + t_p \vec{v}_p$

$\vec{u} + \vec{w} = (s_1 + t_1)\vec{v}_1 + \cdots + (s_p + t_p)\vec{v}_p$ ✓

c) $c\vec{u} = (cs_1)\vec{v}_1 + \cdots + (cs_p)\vec{v}_p$

$\text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\}$ is called <u>the subspace spanned</u> by $\{\vec{v}_1, \cdots, \vec{v}_p\}$.

Special cases:

$\mathbb{R}^n$ is a subspace.

$\{\vec{0}\}$ is a subspace

it is called <u>the zero subspace</u>.

<u>Column space</u>:

<u>Definition</u>: Consider an $m \times n$ matrix A

$$A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$$

$$\underbrace{\text{Col}\, A}_{\text{column space of } A} = \text{Span}\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}.$$

$\vec{a}_j$ is in $\mathbb{R}^m$

$\Rightarrow$ Col A is a subspace of $\underline{\mathbb{R}^m}$

Q: For a given $\vec{b}$ in $\mathbb{R}^m$, is $\vec{b}$ in Col A ?

Method: $\vec{b}$ is in Col A $\Leftrightarrow$ $\vec{b}$ is in span$\{\vec{a}_1, \cdots, \vec{a}_n\}$

$\Leftrightarrow$ $A\vec{x} = \vec{b}$ is consistent

$\Leftrightarrow$ $[A \,\vdots\, \vec{b}]$ has no pivot position in the right most column.

(Theorem 2, Chap 1).

Q: Is Col $A = \mathbb{R}^m$?

Method: Col $A = \mathbb{R}^m$ $\Leftrightarrow$ Span$\{\vec{a}_1, \cdots, \vec{a}_n\} = \mathbb{R}^m$

$\Leftrightarrow$ Matrix A has a pivot position in every row.

(Theorem 4, Chap 1).

Ex $A = \begin{bmatrix} 1 & -3 & 4 \\ -2 & 0 & -4 \\ -3 & 7 & 8 \end{bmatrix}$

Q: Is $\mathrm{Col}\,A = \mathbb{R}^3$

$$A \sim \begin{bmatrix} 1 & -3 & 4 \\ 0 & -6 & -12 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -3 & 4 \\ 0 & \boxed{-6} & -12 \\ 0 & 0 & 0 \end{bmatrix}$$

R3 does not have a pivot position.

$\Rightarrow \mathrm{Col}\,A \neq \mathbb{R}^3$

Null space of A

Definition. A: $m \times n$

$$\underbrace{\mathrm{Nul}\,A}_{\text{Null space of } A} = \left(\text{Solution set of } A\vec{x} = \vec{0}\right)$$

$\vec{x}$ is in $\mathbb{R}^n$.

Nul A is subset of $\mathbb{R}^n$

Verify 3 Conditions

a) $\vec{0}$ is in Nul A: $A\vec{0} = \vec{0}$ ✓

b). $A\vec{u} = \vec{0}$, $A\vec{v} = \vec{0} \Rightarrow A(\vec{u}+\vec{v}) = \vec{0}$

c) $A\vec{u} = \vec{0} \implies A(c\vec{u}) = c(A\vec{u}) = \vec{0}$ ✓

In conclusion. Nul A is a subspace of $\mathbb{R}^n$

Theorem 12 (Chapter 2)

For an $m \times n$ matrix A

Nul A is a subspace of $\mathbb{R}^n$, and

Col A is a subspace of $\mathbb{R}^m$

Q: Is $\vec{u}$ in Nul A?

How to test if $\vec{u}$ is in Nul A?

Method: $\vec{u}$ is in Nul A

$\iff A\vec{u} = \vec{0}$ (verify it directly).

Basis for a subspace

Definition: Let H be a subspace of $\mathbb{R}^n$. $\{\vec{v}_1, \cdots, \vec{v}_p\}$ in H is called a basis for H if two conditions are satisfied

1). $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is linearly independent.

2). Span$\{\vec{v}_1, \cdots, \vec{v}_p\} = H$

Ex. Consider an $n \times n$ matrix

$$A = [\vec{a}_1 \cdots \vec{a}_n] \qquad \vec{a}_j \text{ is in } \mathbb{R}^n$$

Suppose A is invertible.

Claim: $\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$ is a basis for $\mathbb{R}^n$.

proof: ①. Theorem 8 (Chap 2) ⓔ.

$\Longrightarrow \{\vec{a}_1, \cdots, \vec{a}_n\}$ is linearly independent ✓

②. Theorem 8 (Chap 2) ⓗ.

$\Longrightarrow \text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\} = \mathbb{R}^n$ ✓

A special case:

I : $n \times n$ identity matrix

$$I = [\vec{e}_1 \ \vec{e}_2, \cdots, \vec{e}_n] \quad , \quad \vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th entry}$$

$\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$

is called the standard basis for $\mathbb{R}^n$.

$n=3$: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is the standard basis for $\mathbb{R}^3$

Q: How to find a basis for Nul A?

Method: We find the solution set in parametric form

Ex. $A = \begin{bmatrix} \boxed{1} & 3 & 0 & 2 \\ 0 & 0 & \boxed{1} & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ already in reduced echelon form (REF)

Find a basis for Nul A:

We solve $A\vec{x} = \vec{0}$.

Basic variables: x_1, x_3

Free variables: x_2, x_4.

Solution set in parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 - 2x_4 \\ x_2 \\ -5x_4 \\ x_4 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}} + x_4 \underbrace{\begin{bmatrix} -2 \\ 0 \\ -5 \\ 1 \end{bmatrix}}_{\vec{v}}$$

$\vec{x} = x_2 \vec{u} + x_4 \vec{v}$

Claim: $\{\vec{u}, \vec{v}\}$ is a basis for Nul A.

proof: ① $\{\vec{u}, \vec{v}\}$ is linearly independent ✓

$x_2 \vec{u} + x_4 \vec{v} = \vec{0}$

$\Rightarrow x_2 = 0$ and $x_4 = 0$

$\Rightarrow$ linear independence

② $\text{Span}\{\vec{u}, \vec{v}\} = \text{Nul } A$. ✓

In conclusion $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A.

Q: How to find a basis for Col A?

Method: We identify pivot columns.

Ex:

$$A = \begin{array}{c} \begin{bmatrix} \boxed{1} & 3 & 0 & 2 \\ 0 & 0 & \boxed{1} & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \end{array} \end{array}$$

already in REF.

pivot columns: $\vec{a}_1, \vec{a}_3$

Claim: $\{\vec{a}_1, \vec{a}_3\}$ is a basis for Col A.

proof: 1. In REF, each pivot column has one unique non-zero entry.

$$c_1\vec{a}_1 + c_2\vec{a}_3 = 0 \longrightarrow \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ 0 \end{bmatrix} = 0$$

$\Rightarrow c_1 = 0, \; c_2 = 0$

$\Rightarrow$ linear independence. ✓

2. Other columns can be expressed in terms of $\vec{a}_1$ and $\vec{a}_3$

$$\vec{a}_4 = 2\vec{a}_1 + 5\vec{a}_3$$

$$\vec{a}_2 = 3\vec{a}_1$$

$$\text{Span}\{\vec{a}_1, \vec{a}_3\} = \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\} = \text{Col } A.$$

In Conclusion. $\{\vec{a}_1, \vec{a}_3\}$ is a basis for Col A.

Theorem 13 (Chap 2).

The pivot columns of A form a basis for Col A.

procedure for finding a basis for Col A

Step 1: Row reduction to an echelon form.

Step 2: Identify pivot columns.

Step 3: Take the pivot columns of matrix A

proof: EROs do not change the relation among columns.

EROs do change the column space.

$$A \xrightarrow{\text{row reduction}} B.$$

$$\text{Col } A \neq \text{Col } B$$

Ex
$$A = \begin{bmatrix} 1 & 3 & -1 & -3 \\ 2 & 6 & 1 & 9 \\ 1 & 3 & -2 & -8 \end{bmatrix}$$

Find a basis for Col A

$$A \sim \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & 0 & 3 & 15 \\ 0 & 0 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 3 & -1 & -3 \\ 0 & 0 & \boxed{3} & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Col 1 and Col 3 are pivot columns.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\}$$ is a basis for Col A.

In particular

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \right\}$$ is not a basis for Col A

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$$

None of the columns of A has the form $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$