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Recap Matrix multiplication

In general, *) $AB \neq BA$

*) $AB = 0$ does not imply $A=0$ or $B=0$

*) $AB = AC$ does not imply $B = C$

<u>Theorem 2</u> (Chap 2) properties of AB

<u>Transpose</u>: A^T

<u>Theorem 3</u> (Chap 2) properties of A^T

⋮

$$(AB)^T = B^T A^T$$

<u>Inverse</u> of a matrix: A^{-1}

<u>Definition</u>: $A^{-1}A = AA^{-1} = I$

<u>Theorem 6</u> (Chap 2) properties of A^{-1}

⋮

If both A and B are invertible, then AB is invertible.

and $(AB)^{-1} = B^{-1}A^{-1}$

Theorem 5 (Chap 2)

$$A\vec{x}=\vec{b} \xrightarrow{\text{A is invertible}} \vec{x}=A^{-1}\vec{b}$$

That is, invertibility of A implies existence and uniqueness of solution of $A\vec{x}=\vec{b}$

Elementary row operation (ERO)

$\longleftrightarrow$ Elementary matrix (EM)

An ERO on A

$\longleftrightarrow$ Left multiplying A by the corresponding EM

Theorem 7 (Chap 2)

*) Matrix A is invertible if and only if A is row equivalent to I.

*) Suppose $E_p \cdots E_2 E_1 A = I$.

Then $A^{-1} = E_p \cdots E_2 E_1 I$

That is, the sequence of EROs that reduces A to I reduces I to A^{-1}

An algorithm for finding A^{-1}

We do row reduction on $[A \,\vdots\, I]$

If $[A \,\vdots\, I] \xrightarrow{\text{row reduction}} [I \,\vdots\, \square]$

then A is invertible and

$A^{-1} = \square$

Otherwise A is not invertible.

<u>Ex</u>. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & -1 & 6 \end{bmatrix}$

$$[A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 4 & -1 & 6 & 0 & 0 & 1 \end{array}\right]$$

Interchange R1 and R2

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -1 & 6 & 0 & 0 & 1 \end{array}\right]$$

Add $(-4) \times R1$ to R3

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -6 & 0 & -4 & 1 \end{array}\right]$$

Add $5 \times R2$ to $R3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 5 & -4 & 1 \end{array}\right]$$

Multiply $R3$ by $1/4$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5/4 & -1 & 1/4 \end{array}\right]$$

Add $(-2) \times R3$ to $R2$

Add $(-3) \times R3$ to $R1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -15/4 & 4 & -3/4 \\ 0 & 1 & 0 & -3/2 & 2 & -1/2 \\ 0 & 0 & 1 & 5/4 & -1 & 1/4 \end{array}\right]$$

Add $(-1) \times R2$ to $R1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/4 & 2 & -1/4 \\ 0 & 1 & 0 & -3/2 & 2 & -1/2 \\ 0 & 0 & 1 & 5/4 & -1 & 1/4 \end{array}\right]$$

$\Longrightarrow$ A is invertible.

$$A^{-1} = \begin{bmatrix} -9/4 & 2 & -1/4 \\ -3/2 & 2 & -1/2 \\ 5/4 & -1 & 1/4 \end{bmatrix}$$

Let us verify

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & -1 & 6 \end{bmatrix} \begin{bmatrix} -9/4 & 2 & -1/4 \\ -3/2 & 2 & -1/2 \\ 5/4 & -1 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} \overbrace{1\times(-3/2)+2\times 5/4}^{=1} & \overbrace{1\times 2+2\times(-1)}^{=0} & \overbrace{1\times(-1/2)+2\times 1/4}^{=0} \\ \underbrace{-9/4-3/2+3\times 5/4}_{=0} & \underbrace{2+2-3}_{=1} & \underbrace{-1/4-1/2+3/4}_{=0} \\ \underbrace{\underbrace{4\times(-9/4)+3/2+6\times\frac{5}{4}}_{-9+3/2+\frac{15}{2}}}_{=0} & \underbrace{4\times 2-2-6}_{=0} & \underbrace{4\times(-1/4)+1/2+6\times 1/4}_{=1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We only need to check $AA^{-1} = I$

or $A^{-1}A = I$

<u>Statement.</u> To verify that A is invertible and $A^{-1} = B$, we only need to verify.

$$AB = I \quad \text{or} \quad BA = I$$

Revisit $AB = AC$

Q: What happens if A is invertible?

*) $AB \neq BA$

*) $AB = AC$ implies $B = C$

proof: $A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$

$\Rightarrow I \cdot B = I \cdot C$

$\Rightarrow B = C$

Sec 2.3. Characterization of invertible matrices.

Theorem 8 (The invertible matrix theorem)

Let A be an $n \times n$ matrix.

Then all statements below are equivalent to each other.

a) A is invertible.

b). A is row equivalent to I.

b) $\Longleftrightarrow$ a) by Theorem 7 (Chap 2)

c) A has n pivot positions
(A has a pivot position in every row)

c) $\Longleftrightarrow$ b).

<u>proof</u>: n pivot positions $\Longleftrightarrow$ reduced echelon form $= I$

d) $A\vec{x}=\vec{0}$ has only the trivial solution.

d) $\Longleftrightarrow$ c).

<u>proof</u>: Only the trivial solution
$\Longleftrightarrow$ solution is unique.
$\Longleftrightarrow$ no free variable (<u>Theorem 2</u>, chap 1).
$\Longleftrightarrow$ n pivot positions.

e) The columns of A form a linearly independent set.

e) $\Longleftrightarrow$ d.
(definition of linear independence).

f). <u>skip</u>

g). $A\vec{x}=\vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^n$.

g) $\Longleftrightarrow$ c).

<u>proof</u>: <u>Theorem 4</u> (chap 1).

h). The columns of A span $\mathbb{R}^n$.

h) $\Longleftrightarrow$ g) $\Longleftrightarrow$ c).

<u>proof</u> <u>Theorem 4</u> (chap 1).

i) Skip

j) There is an $n \times n$ matrix C such that

$$CA = I$$

j) $\Longleftrightarrow$ a).

proof: "a) $\Longrightarrow$ j)" definition ✓

"j) $\Longrightarrow$ a)"

We only need to show "j) $\Longrightarrow$ d)"

$$A\vec{x} = \vec{0}.$$

$$\to C(A\vec{x}) = C\vec{0}$$

$$\to (CA)\vec{x} = \vec{0}$$

$$\to I\vec{x} = \vec{0}$$

$$\to \vec{x} = \vec{0}.$$ only the trivial solution.

k). There is an $n \times n$ matrix D such that

$$AD = I$$

k) $\Longleftrightarrow$ a)

proof. "a) $\Longrightarrow$ k)" definition ✓

"k) $\Longrightarrow$ a)"

We only need to show k) $\Longrightarrow$ g)

We need to write out a solution of $A\vec{x} = \vec{b}$

Try $\vec{x} = D\vec{b}$

$$A\vec{x} = A(D\vec{b}) = (AD)\vec{b} = I\vec{b} = \vec{b}$$

e). A^T is invertible.

e) $\Longleftrightarrow$ a).

proof: Theorem 6 (Chap 2).

B is invertible $\Longrightarrow$ B^T is invertible.

let $B = A^T$.

$B = A^T$ is invertible $\Longrightarrow$ $B^T = (A^T)^T = A$ is invertible

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Theorem 8 Supplemented

Let A and B be $n \times n$ matrices.

If $AB = I$, then both A and B are invertible.

and $A^{-1} = B$

$B^{-1} = A$

Theorem 8 c) gives us a simple way of checking if a matrix is invertible.

Ex. $A = \begin{bmatrix} 1 & 4 & -4 \\ 1 & 3 & -2 \\ -1 & -5 & 9 \end{bmatrix}$

Add $(-1) \times R1$ to $R2$
Add $R1$ to $R3$

$$\begin{bmatrix} 1 & 4 & -4 \\ 0 & -1 & 2 \\ 0 & -1 & 5 \end{bmatrix}$$

Add $(-1) \times R2$ to $R3$

$$\begin{bmatrix} \boxed{1} & 4 & -4 \\ 0 & \boxed{-1} & 2 \\ 0 & 0 & \boxed{3} \end{bmatrix}$$

3 pivot positions

$\longrightarrow$ A is invertible

Q: How to find one column of A^{-1}.

$$A A^{-1} = I.$$

We write $I = [\vec{e}_1 \ \vec{e}_2 \ \cdots \ \vec{e}_n]$

$$\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th entry}$$

$$A^{-1} = [\vec{u}_1 \ \vec{u}_2 \ \cdots \ \vec{u}_n]$$

$$A\,[\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_n] = [\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n]$$

$$A\vec{u}_j = \vec{e}_j$$

To calculate $\vec{u}_j$, we just solve $A\vec{u}_j = \vec{e}_j$

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problem 10 (homework 5)

Suppose A, B and $(A+B)$ are all invertible.

Show that $(A^{-1}+B^{-1})$ is invertible and

$$(A^{-1}+B^{-1})^{-1} = A(A+B)^{-1}B$$

Strategy: To show P is invertible, ~~we~~

*) we only need to show that matrix Q (a candidate for P^{-1}) is invertible and $Q^{-1} = P$

*) We only need to show

$$QP = I \quad \text{or} \quad PQ = I.$$

$$P = (A^{-1}+B^{-1})$$

$$Q = A(A+B)^{-1}B$$

Q is invertible. (because A, B and $(A+B)$ are all invertible)

$$Q^{-1} = \left(A\,(A+B)^{-1} B \right)^{-1}$$
$$= B^{-1} \left((A+B)^{-1} \right)^{-1} A^{-1}$$
$$= B^{-1} (A+B) A^{-1}$$
$$= B^{-1} A A^{-1} + B^{-1} B A^{-1}$$
$$= B^{-1} + A^{-1} = P$$

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problem 7 (homework 5)

A : $n \times n$

B : $n \times n$.

Suppose AB is invertible.

Show that both A and B are invertible.

proof: (AB) is invertible.

$\Rightarrow$ There exists Q such that (statement (k) in Theorem 8)

$(AB)\,Q = I$

$\Rightarrow$ $A(BQ) = I$.

$\Rightarrow$ $AD = I$ $\quad D = BQ$

$\Rightarrow$ $\boxed{A \text{ is invertible}}$. (statement (k) in Theorem 8)

(AB) is invertible

$\Longrightarrow$ There exists P such that

$P(AB) = I$ (Statement j in Theorem 8)

$\Longrightarrow (PA)B = I$

$\Longrightarrow CB = I$, $\underline{C = PA}$

$\Longrightarrow$ $\boxed{B \text{ is invertible}}$ (Statement j in Theorem 8).