02/01/2018

Recap

Q: Is span$\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\} = \mathbb{R}^m$?

Q: Does $A\vec{x} = 0$ have a non-trivial solution?

Q: Is $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ linearly independent?

Q: Is $A\vec{x} = \vec{b}$ consistent for the given $\vec{b}$?

Q: Is $A\vec{x} = \vec{b}$ consistent for every $\vec{b}$?

Theorem 7:

A set of vectors is linearly dependent iff one vector is a linear combination of the others.

3 special cases:

1) A set of 2 vectors
2) A set containing $\vec{0}$ (Theorem 9)
3) # of vectors in the set > # of entries of each vector (Theorem 8)

Matrix multiplication (Chapter 2)

A: $m \times n$

B: $n \times p$

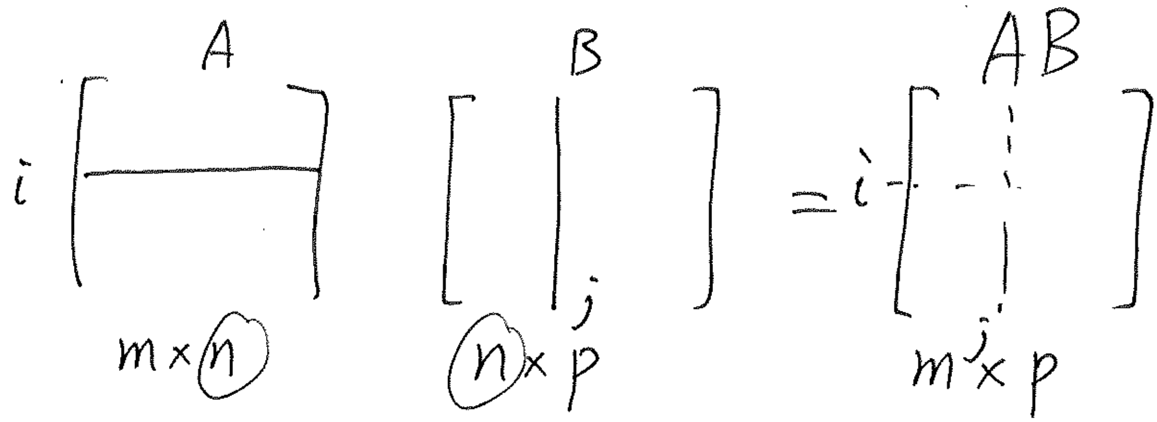
$B = [\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_p]$

$AB = [A\vec{b}_1 \ A\vec{b}_2 \cdots A\vec{b}_p]$

AB is defined only when
# of columns in A
= # of rows in B

The row–column rule (for calculating AB)

$$(AB)_{ij} = \text{row}_i(A) \cdot \text{Col}_j(B)$$



Perculiar properties of AB

*) In general $AB \neq BA$

Ex.

$$\begin{array}{ccc} A & B & AB \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} B & A & BA \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$AB \neq BA$$

$BA = B\,0_{2\times 2}$

*) $BA = 0$ does not imply $A = 0$ or $B = 0$.

*). $BA = BC$ does not imply $A = C$

Theorem 2 (properties of AB)

a. $A(BC) = (AB)C$

b. $A(B+C) = AB + AC$

c. $(B+C)A = BA + CA$

d. $r(AB) = (rA)B = A(rB)$

e. A: $m \times n$

$$I_m A = A I_n = A$$

Proof of a. We first show $A(B\vec{x}) = (AB)\vec{x}$ ✓

A: $m \times n$.

B: $n \times p$ $\quad B = [\vec{b_1}\ \vec{b_2} \cdots \vec{b_p}]$

$\vec{x}$: $p \times 1$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$

$$B\vec{x} = x_1 \vec{b_1} + \cdots + x_p \vec{b_p}$$

$$AB = [A\vec{b_1} \cdots A\vec{b_p}]$$

$$A(B\vec{x}) = A(x_1\vec{b}_1 + \cdots + x_p\vec{b}_p)$$
$$= x_1(A\vec{b}_1) + \cdots + x_p(A\vec{b}_p)$$
$$= [(A\vec{b}_1) \cdots (A\vec{b}_p)]\,\vec{x}$$
$$= (AB)\vec{x}$$

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let $C = [\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_k]$

$$A(BC) = A[B\vec{c}_1\ \ B\vec{c}_2 \cdots B\vec{c}_k]$$
$$= [A(B\vec{c}_1)\ \ A(B\vec{c}_2) \cdots A(B\vec{c}_k)]$$
$$= [(AB)\vec{c}_1\ \ (AB)\vec{c}_2 \cdots (AB)\vec{c}_k]$$
$$= (AB)C$$

Powers of a square matrix

A : $n \times n$.

$A^2 = AA$

$A^3 = A^2 A$

$A^k = \underbrace{A\,A \cdots A}_{k}$.

$A^0 = I$

$$\underset{n\times n}{\overset{A}{[\quad]}}\ \underset{n\times n}{\overset{A}{[\quad]}} = \underset{n\times n}{\overset{A^2}{[\quad]}}$$

Transpose of a matrix

$A : m \times n$

$$A = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \Big\} \, m \text{ rows}$$

A^T : transpose of A. : $n \times m$

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & & a_{mn} \end{bmatrix}$$

$$\boxed{(A^T)_{ij} = (A)_{ji}}$$

$$\text{row}_i(A^T) = \text{col}_i(A)$$

$$\text{col}_j(A^T) = \text{row}_j(A)$$

Ex. $A = \begin{bmatrix} 2 & 3 & -4 \\ -3 & 1 & 5 \end{bmatrix}$: 2×3

$A^T : 3 \times 2$ $\qquad A^T = \begin{bmatrix} 2 & -3 \\ 3 & 1 \\ -4 & 5 \end{bmatrix}$

Theorem 3 (properties of A^T)

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. $(rA)^T = r A^T$

d. $(AB)^T = B^T A^T$

proof of (d)

We first show

$$\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \cdots + a_n b_n$$

$$= b_1 a_1 + \cdots + b_n a_n$$

$$= \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

We show $\left((AB)^T\right)_{ij} = \left(B^T A^T\right)_{ij}$

$$\left((AB)^T\right)_{ij} = (AB)_{ji}$$

$$= \text{row}_j(A) \cdot \text{col}_i(B)$$

$$= \text{row}_i(B^T) \cdot \text{col}_j(A^T) = \left(B^T A^T\right)_{ij}$$

$$(A_1 A_2 \cdots A_K)^T = A_K^T A_{K-1}^T \cdots A_1^T$$

Sec 2.2 The inverse of a matrix

Recall the inverse of a number

a^{-1} satisfies $a^{-1} a = 1$

Def: Consider A : $n \times n$

A is said to be invertible if there is an $n \times n$ matrix C

such that

$$CA = I_n \quad \text{and} \quad AC = I_n$$

Results and notations:

*) Matrix C, if exists, is unique.

· Suppose B also satisfies

$$BA = I_n \quad \text{and} \quad AB = I_n.$$

$$B = BI = B(AC) = (BA)C = IC = C$$

*). C is called the inverse of matrix A

denoted by A^{-1}

A^{-1} satisfies $A^{-1}A = I_n \quad AA^{-1} = I$

*). An invertible matrix is also called non-singular.

Theorem 5: If A is invertible, then

$A\vec{x}=\vec{b}$ has the unique solution $\vec{x}=A^{-1}\vec{b}$

proof $A\vec{x}=\vec{b}$

$\rightarrow A^{-1}(A\vec{x})=A^{-1}\vec{b}$

$\rightarrow (A^{-1}A)\vec{x}=A^{-1}\vec{b}$

$\rightarrow I\vec{x}=A^{-1}\vec{b}$

$\rightarrow \vec{x}=A^{-1}\vec{b}$

Theorem 6 (properties of A^{-1}).

(a) If A is invertible, then A^{-1} is also invertible and $(A^{-1})^{-1}=A$ ✓

proof: let $C=A$

$CA^{-1}=AA^{-1}=I$

$A^{-1}C=A^{-1}A=I$

(b). If A and B are both invertible, then (AB) is invertible and

$(AB)^{-1}=B^{-1}A^{-1}$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$

If $A_1, A_2, \cdots, A_K$ are all invertible,

the $(A_1 A_2 \cdots A_K)$ is invertible and

$$(A_1 A_2 \cdots A_K)^{-1} = A_K^{-1} \cdots A_1^{-1}$$

ⓒ If A is invertible, the A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Elementary matrices

Def: The result of performing an ERO on the identity matrix is called an *elementary matrix*

Ex:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{Add } (-3) \times R1 \text{ to } R3}$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

E_1 is an elementary matrix

Effect of left-multiplying matrix A by E_1

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$E_1 A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ -3a_{11}+a_{31} & -3a_{12}+a_{32} & -3a_{13}+a_{33} & -3a_{14}+a_{34} \end{bmatrix}$$

Statement: Performing an ERO on A

= left-multiplying A by the corresponding elementary matrix.

Statement: Every elementary matrix is invertible.

Ex. $$(E_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Theorem 7:

*) Matrix A is invertible iff A is row equivalent to I.

proof: "if"

$$E_p \cdots E_2 E_1 A = I \quad \left(\begin{array}{c} A \text{ is row equivalent} \\ \text{to } I \end{array}\right)$$

Each of $E_1, E_2, \cdots, E_p$ is invertible.

$\Rightarrow$ $(E_p \cdots E_2 E_1)$ is invertible.

$$\underbrace{(E_p \cdots E_2 E_1)^{-1}(E_p \cdots E_1 A)}_{\displaystyle \| \atop \displaystyle \underbrace{\left((E_p \cdots E_1)^{-1}(E_p \cdots E_1)\right) A}_{\displaystyle \| \atop \displaystyle A}} = (E_p \cdots E_1)^{-1}$$

$\Rightarrow$ $A = (E_p \cdots E_1)^{-1}$

$\Rightarrow$ A is invertible and $A^{-1} = (E_p \cdots E_1) I$

<u>proof</u> "only if"

A is invertible.

$\Rightarrow$ $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$.

$\Rightarrow$ A has pivot position in every row.

$\Rightarrow$ The reduced echelon form of A is identity matrix I.

$\Rightarrow$ A is row equivalent to I.

*) Suppose $E_p \cdots E_1 A = I$.

Then we have $A^{-1} = (E_p \cdots E_1) I$

An algorithm for finding A^{-1}

We do row reduction on $[A \mid I]$

$$[A \mid I] \xrightarrow{\text{row reduction}} [I \mid \square]$$

$\square$: A^{-1}

If the left half is I, then A is invertible and

$$A^{-1} = \square$$