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$\boxed{\text{Recap}}$ Q: Is $A\vec{x}=\vec{b}$ consistent for every $\vec{b}$?

Theorem 4: A: $m\times n$, $A=[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$

4 statements are equivalent.

a. $A\vec{x}=\vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^m$

b. Every $\vec{b}$ in $\mathbb{R}^m$ is in span$\{\vec{a}_1,\cdots,\vec{a}_n\}$

c. span$\{\vec{a}_1,\cdots,\vec{a}_n\}=\mathbb{R}^m$

d. Matrix A has a pivot position in every row

not the augmented matrix.

Dot product of a row and a vector

Row-vector rule for calculating $A\vec{x}$

$n\times n$ identity matrix

$$I=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 1\end{bmatrix}$$

$I\vec{x}=\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$

Theorem 5 (Properties of $A\vec{x}$)

*) $A(\vec{u}+\vec{v})=?$ *) $A(c\vec{u})=?$ *) $I\vec{x}=?$ *) $A\vec{0}=?$

Structure of the solution set

Homogeneous linear system: $A\vec{x}=\vec{0}$

*) always consistent

*) $\vec{x}=\vec{0}$ is called the trivial solution.

Statement: $A\vec{x}=\vec{0}$ has a non-trivial solution if and only if it has at least one free variable.

Solution set in parametric form

Ex.
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Solution set} = \text{Span}\left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Non-homogeneous system: $A\vec{x}=\vec{b}$

*) is consistent for a given $\vec{b}$ if and only if the rightmost column of $[A \mid \vec{b}]$ is NOT a pivot column.

*) is consistent for every $\vec{b}$ if and only if matrix A has a pivot position in every row

Theorem 6: Suppose $A\vec{x}=\vec{b}$ has a solution $\vec{p}$

Solution set of $A\vec{x}=\vec{b}$

$= \vec{p}$ + Solution set of $A\vec{x}=\vec{0}$

Linear independence / linear dependence

Review of theorems we learned.

Theorem 1:

Matrix $\xrightarrow{\text{row reduction}}$ reduced echelon form

Theorem 2:

Existence ? (is the right most column is a pivot column?)

Uniqueness ? (is there any free variable ?

Theorem 3:

Equivalence of matrix eq, vector eq, linear system)

Theorem 4:

a)

b)

c)

d) Matrix A has a pivot position in every row.

Theorem 5. (properties of $A\vec{x}$)

Theorem 6: Solution set of $A\vec{x} = \vec{b}$

$= \vec{p}$ + Solution set of $A\vec{x} = \vec{0}$

$\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_p\}$ is linearly dependent

if and only if $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$

has a non-trivial solution.

Statement: Columns of matrix A are linearly dependent if and only if $A\vec{x} = \vec{0}$ has a non-trivial solution.

Q: Is span $\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\} = \mathbb{R}^m$? $\vec{a}_j$ is in $\mathbb{R}^m$

Method: Check pivot positions of $A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$ (Theorem 4)

Q: Does $A\vec{x} = \vec{0}$ has a non-trivial solution?

Method: Check free variable(s). (Theorem 2)

Q: Is $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ Linearly independent?

$Q \Longleftrightarrow A\vec{x} = \vec{0}$ has only the trivial solution?

where $A = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_p]$.

Method: Check free variable.

Ex. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Q1: Are they linearly dependent?

Q2: If so, find a linear dependence relation.

$$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 2 & 2 & -2 \end{bmatrix}$$

$$[A \,\vdots\, \vec{0}] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 2 & 2 & -2 & 0 \end{array}\right]$$

$\xrightarrow{\text{echelon form}}$ Add $(-2) \times R1$ to $R2$
Add $(-2) \times R1$ to $R3$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right]$$

Add $(-2) \times R2$ to $R3$.

$$\to \left[\begin{array}{ccc|c} \boxed{1} & 4 & 2 & 0 \\ 0 & \boxed{-3} & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variables: x_1, x_2

Free variable: x_3

$\Rightarrow$ $A\vec{x} = \vec{0}$ has a non-trivial solution

$\Rightarrow$ $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

reduced echelon form $\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$ solution set in parametric form.

pick $x_3 = 2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

$$\boxed{4\vec{v}_1 - 2\vec{v}_2 + 2\vec{v}_3 = 0}$$

Ex. $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 2 & 2 & -4 \end{bmatrix}$

Q: Are the columns of A linearly dependent.

Q $\Longleftrightarrow$ Does $A\vec{x} = \vec{0}$ has a non-trivial solution?

$$[A \,|\, 0] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right]$$

row reduction $\longrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 4 & 2 & 0 \\ 0 & \boxed{-3} & -3 & 0 \\ 0 & 0 & \boxed{-2} & 0 \end{array}\right]$

$\Rightarrow$ No free variable.

$\Rightarrow$. Columns of A are linearly independent..

Consequence of linear dependence.

A set of one vector.

$\{\vec{v}\}$. is linearly dependent

if and only if $\quad c\vec{v} = \vec{0}$. and $c \neq 0$.

$\iff \vec{v} = \vec{0}$

A set of p vectors $(p \geq 2)$.

$\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ is linearly dependent

iff $\quad c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$

and not all $c_1, c_2, \cdots, c_p$ are zeros.

Suppose $c_p \neq 0$.

$$v_p = -\left(\frac{c_1}{c_p}\right)\vec{v}_1 - \left(\frac{c_2}{c_p}\right)\vec{v}_2 - \cdots - \left(\frac{c_{p-1}}{c_p}\right)\vec{v}_{p-1}$$

Theorem 7: $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ is linearly dependent iff

One vector is a linear combination of others.

3 special cases:

$\boxed{\text{Case 1}}$: A set of two vectors.

$\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent iff one is a multiple of the other.

Ex. $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ are linearly independent.

$\boxed{\text{Case 2}}$: A set containing the zero vector.

$$\vec{0} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots$$

Theorem 9: If a set contains the zero vector, then the set is linearly dependent.

Ex $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is linearly dependent.

$\boxed{\text{Case 3}}$: Theorem 8: $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ in $\mathbb{R}^m$ is linearly dependent if $\underline{p > m}$.

proof. $A = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_p]$ $\quad A: m \times p$.

Consider $\underset{m\times p}{A}\ \underset{p\times 1}{\vec{x}} = \vec{0}$.

A has m rows

$\Rightarrow$ # of pivot positions $\leq m$

$\Rightarrow$ # of basic variables $\leq m$

Total # of variables $= p. > m$

$\Rightarrow$ # of free variables $= p -$ # of basic variables

$\geq p - m > 0$

$\Rightarrow A\vec{x} = \vec{0}$ has a non-trivial solution.

$\Rightarrow \{\vec{v}_1, \cdots, \vec{v}_p\}$ is linearly dependent.

Ex. $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

2 entries in each vector

3 vectors in the set

$3 > 2$

$\rightarrow$ The set is linearly dependent.

Sec 2.1 Matrix operations.

A: $m \times n$.

$A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$, $\vec{a}_j$ is in $\mathbb{R}^m$

B: $m \times n$

$B = [\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_n]$, $\vec{b}_j$ is in $\mathbb{R}^m$

Addition: $A+B = [(\vec{a}_1+\vec{b}_1) \ (\vec{a}_2+\vec{b}_2) \cdots (\vec{a}_n+\vec{b}_n)]$

Caution: $A+B$ is defined only when A and B are of the same size.

Scalar multiplication:

$$rA = [r\vec{a}_1 \ r\vec{a}_2 \cdots r\vec{a}_n]$$

Ex. $A = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 4 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 7 \\ 6 & 3 \end{bmatrix}$

$$A+B = \begin{bmatrix} 1 & 5 & 11 \\ 4 & 9 & 7 \end{bmatrix}$$

$A+C =$ not defined!

$$-A = (-1)A$$

$$A - B = A + (-1)B$$

Ex. $A - 2B = \begin{bmatrix} -1-2\times 2 & 2-2\times 3 & 4-2\times 7 \\ 3-2\times 1 & 5-2\times 4 & 1-2\times 6 \end{bmatrix}$

$$= \begin{bmatrix} -5 & -4 & -10 \\ 1 & -3 & -11 \end{bmatrix}$$

The zero matrix.

$$O_{m\times n} = \underbrace{\begin{bmatrix} \vec{0} & \vec{0} & \cdots & \vec{0} \end{bmatrix}}_{n \text{ columns.}} \Big\} \ m \text{ rows}$$

<u>Theorem 1</u>:

a. $A + B = B + A$

b. $(A+B) + C = A + (B+C)$

c. $A + 0 = A$

d. $r(A+B) = rA + rB$

e. $(r+s)A = rA + sA$

f. $r(sA) = (r\cdot s)A$

Matrix multiplication.

A: $m \times n$

B: $n \times p$.

$B = [\vec{b}_1 \ \vec{b}_2 \cdots \vec{b}_p]$, $\vec{b}_j$ is in $\mathbb{R}^n$

$\underset{m \times n}{A} \ \underset{n \times 1}{b_j}$ is well defined.

Ab_j is in $\mathbb{R}^m$.

AB is defined as

$$AB = \underbrace{[A\vec{b}_1 \ A\vec{b}_2 \cdots A\vec{b}_p]}_{p \text{ columns}} \} \ m \text{ rows}.$$

AB: $m \times p$.

$$\underbrace{A}_{m \times n} \times \underbrace{B}_{n \times p} = \underbrace{AB}_{m \times p}.$$

Caution: $A \cdot B$ is defined only when.
# of columns in A = # of rows in B.

Row-vector rule for $A\vec{x}$

The ith entry of $A\vec{x}$

= dot product of the ith row of A and $\vec{x}$

Row–column rule for calculating AB.

The (i,j) entry of AB

= dot product of the ith row of A and the jth column of B.

Notation $$(AB)_{ij} = \text{row}_i(A) \cdot \text{col}_j(B)$$

Ex $\underbrace{A = \begin{bmatrix} 1 & 5 \\ 3 & -2 \end{bmatrix}}_{2 \times 2}$, $\underbrace{B = \begin{bmatrix} 3 & 2 & 1 \\ -2 & 4 & 6 \end{bmatrix}}_{2 \times 3}$

$$\underbrace{AB}_{2\times 3} = \begin{bmatrix} 1\times 3 + 5\times(-2) & 1\times 2 + 5\times 4 & 1\times 1 + 5\times 6 \\ 3\times 3 + (-2)\times(-2) & 3\times 2 + (-2)\times 4 & 3\times 1 + (-2)\times 6 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 22 & 31 \\ 13 & -2 & -9 \end{bmatrix}$$

$\underbrace{B}_{2\times 3}\underbrace{A}_{2\times 2}$ = not defined.

$\underbrace{B}_{2\times 3} \cdot \underbrace{B}_{2\times 3}$ = not defined.