AMS 10/10A, Homework 4

Problems for Section 1.5 and 1.7

Problem 1. For each of the linear systems below, write the solution set in parametric vector form.

$$\begin{cases} x_1 + 2x_2 &= 0 \\ 3x_1 + 6x_2 &= 0 \\ 5x_1 + 10x_2 &= 0 \end{cases}$$

$$\begin{cases} x_1 + 3x_2 + 2x_3 + 4x_4 &= 0 \\ -x_1 + 2x_2 + 3x_3 + x_4 &= 0 \\ 2x_1 - 3x_2 - 5x_3 - x_4 &= 0 \\ x_1 + x_2 \qquad\quad + 2x_4 &= 0 \end{cases}$$

Problem 2. Let A be an $m \times n$ matrix and let vector v be a solution of the equation $Ax = 0$. Show that for any scalar c, the vector $c\,v$ is also a solution of $Ax = 0$.

Problem 3. Let $\{v_1,\ v_2,\ \cdots,\ v_n\}$ be a set of vectors in $\mathbb{R}^m$ such that v_n is a linear combination of $\{v_1,\ v_2,\ \cdots,\ v_{n-1}\}$. Show that homogeneous equation $Ax = 0$ has infinitely many solutions, where matrix $A = [v_1, v_2, \cdots, v_n]$.

Problem 4. Let A be a matrix whose echelon form is

$$\begin{bmatrix} -2 & -3 & 1 & 4 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

Does $Ax = b$ has a solution for every $b \in \mathbb{R}^3$? Explain.

Problem 5. Mark each statement True or False

5.1. The equation $Ax = b$ is homogeneous if the zero vector is a solution.

5.2. The homogeneous equation $Ax = 0$ has the trivial solution if and only if the equation has at least one free variable.

5.3. A homogeneous system of equations can be inconsistent.

5.4. If v is a nontrivial solution of $Ax = 0$, then every entry in v is nonzero.

5.5. If homogeneous equation $Ax = 0$ has a unique solution, then $Ax = b$ cannot have infinitely many solutions.

Problem 6. Determine if the set of vectors $S = \{v_1, v_2, v_3, v_4\}$ given below is linearly independent, where

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 6 \\ 3 \\ 9 \\ 3 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad v_4 = \begin{bmatrix} 5 \\ 2 \\ 6 \\ 3 \end{bmatrix}.$$

Problem 7. Let $v_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$.

- Show that $\{v_1, v_2, v_3\}$ is linearly dependent.
- Write v_2 as a linear combination of v_1 and v_3.

Problem 8. For what value(s) of k are the columns of the following matrix linearly dependent.

$$\begin{bmatrix} 1 & -5 & 3 \\ 3 & -8 & -5 \\ -1 & 2 & k \end{bmatrix}$$

Problem 9. Let $\{v_1, v_2, v_3\}$ be a set of linearly independent vectors in $\mathbb{R}^n$. Show that $\{v_1 + v_3, v_1 - 2v_2, -4v_1 + v_2 + 3v_3\}$ is also linearly independent.

Problem 10. Mark each statement True or False

10.1. The set $\{0, v_1, v_2, \cdots, v_k\}$ is always linearly dependent.

10.2. Let v_1, v_2, v_3 and v_4 be vectors in $\mathbb{R}^n$ such that $v_1 - v_2 = v_3 - v_4$. Then the the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

10.3. If u and v are linearly independent, and if w is in $span\{u, v\}$, then $\{u, v, w\}$ is linearly dependent.

10.4. If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more than n vectors.

10.5. If $\{v_1, v_2, v_3, v_4\}$ is a set of vectors in $\mathbb{R}^4$ and $\{v_1, v_2, v_3\}$ is linearly dependent, then $\{v_1, v_2, v_3, v_4\}$ is also linearly dependent.

Problems for Section 2.1

Problem 11. Given the following matrices

$$A = \begin{bmatrix} -1 & 5 & 2 \\ 3 & -4 & -1 \\ 2 & 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \\ 7 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & -1 & 2 \\ -3 & 2 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Compute AB, $A^T B$, BC, CD and $(CD)^2$.

Problem 12. How many rows does B have if BC is a 6×8 matrix?

Problem 13. Suppose the second column of B is all zeros. What can be said about the second column of AB?

Problem 14. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix}$. What value(s) of k, if any, will make $AB = BA$?

Problem 15. Prove that if the columns of matrix B are linear dependent, then so are the columns of AB.

Problem 16. Let

$$W = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ 3 & -4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 & 2 \\ 3 & -4 & -1 \end{bmatrix}$$

Use your knowledge of row-column rule to compute $w_{2,1}$, the entry in the 2nd row and 1st column of W without computing the whole product.

Problem 17. Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. Compute $A^2 - 3A$.

Problem 18. Let $B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Show that $B^n = 0$ for all nature number $n \geq 3$.