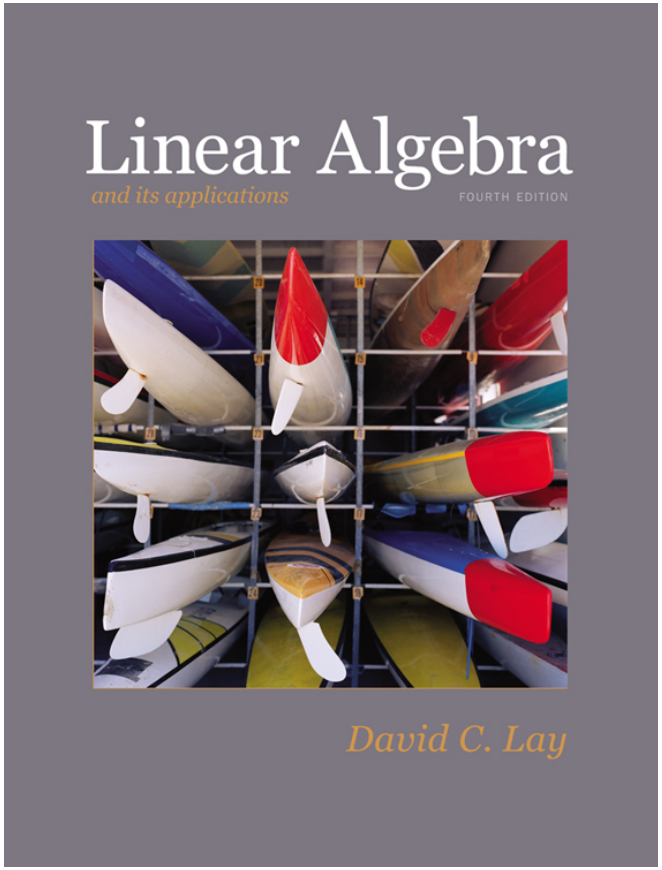
1 Linear Equations in Linear Algebra

1.5

SOLUTION SETS OF LINEAR SYSTEMS



PEARSON

HOMOGENEOUS LINEAR SYSTEMS

- A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.
- Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$).
- This zero solution is usually called the **trivial solution**.
- The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

HOMOGENEOUS LINEAR SYSTEMS

- **Example 1:** Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

- **Solution:** Let A be the matrix of coefficients of the system and row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to echelon form:

HOMOGENEOUS LINEAR SYSTEMS

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Since x_3 is a free variable, $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions (one for each choice of x_3.)
- Continue the row reduction of $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ to *reduced* echelon form:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - \frac{4}{3}x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

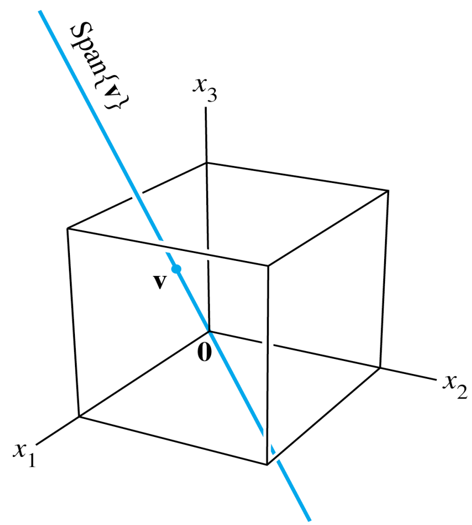
HOMOGENEOUS LINEAR SYSTEMS

- Solve for the basic variables x_1 and x_2 to obtain $x_1 = \frac{4}{3}x_3$, $x_2 = 0$, with x_3 free.
- As a vector, the general solution of $A\mathbf{x} = \mathbf{0}$ has the form given below.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathbf{v}, \text{ where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

HOMOGENEOUS LINEAR SYSTEMS

- Here x_3 is factored out of the expression for the general solution vector.
- This shows that every solution of $A\mathbf{x} = \mathbf{0}$ in this case is a scalar multiple of **v**.
- The trivial solution is obtained by choosing $x_3 = 0$.
- Geometrically, the solution set is a line through **0** in $\mathbb{R}^3$. See the figure below.



PARAMETRIC VECTOR FORM

- The equation of the form $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (s, t in $\mathbb{R}$) is called a **parametric vector equation** of the plane.

- In Example 1, the equation $\mathbf{x} = x_3\mathbf{v}$ (with x_3 free), or $\mathbf{x} = t\mathbf{v}$ (with t in $\mathbb{R}$), is a parametric vector equation of a line.

- Whenever a solution set is described explicitly with vectors as in Example 1, we say that the solution is in **parametric vector form**.

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

- **Example 2:** Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}.$$

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- **Solution:** Row operations on $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ produce

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} x_1 - \frac{4}{3}x_3 &= -1 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}.$$

- Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free.

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form given below.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\mathbf{p}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}}$$

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- The equation $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter,

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \qquad \text{----(1)}$$

  describes the solution set of $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

- The solution set of $A\mathbf{x} = \mathbf{0}$ has the parametric vector equation

$$\mathbf{x} = t\mathbf{v} \qquad (t \text{ in } \mathbb{R}) \qquad \text{----(2)}$$

  [with the same **v** that appears in (1)].

- Thus the solutions of $A\mathbf{x} = \mathbf{b}$ are obtained by adding the vector **p** to the solutions of $A\mathbf{x} = \mathbf{0}$.

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

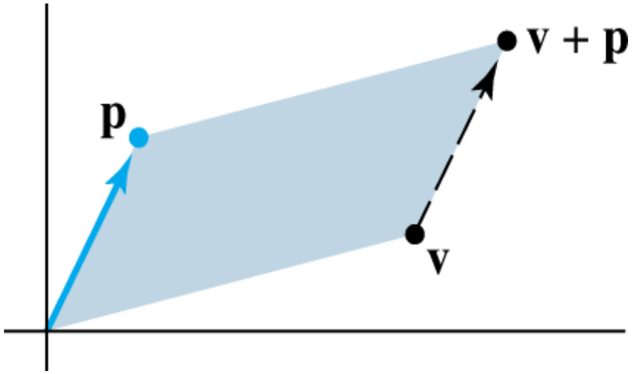
- The vector **p** itself is just one particular solution of $A\mathrm{x} = \mathrm{b}$ [corresponding to $t = 0$ in (1).]

- Now, to describe the solution of $A\mathrm{x} = \mathrm{b}$ geometrically, we can think of vector addition as a *translation*.

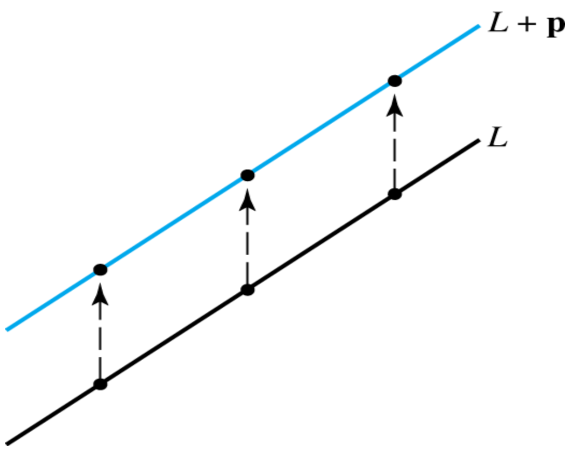
- Given **v** and **p** in $\mathbb{R}^2$ or $\mathbb{R}^3$, the effect of adding **p** to **v** is to *move* **v** in a direction parallel to the line through **p** and **0**.

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- We say that **v** is **translated by p** to $\mathbf{v}+\mathbf{p}$. See the following figure.



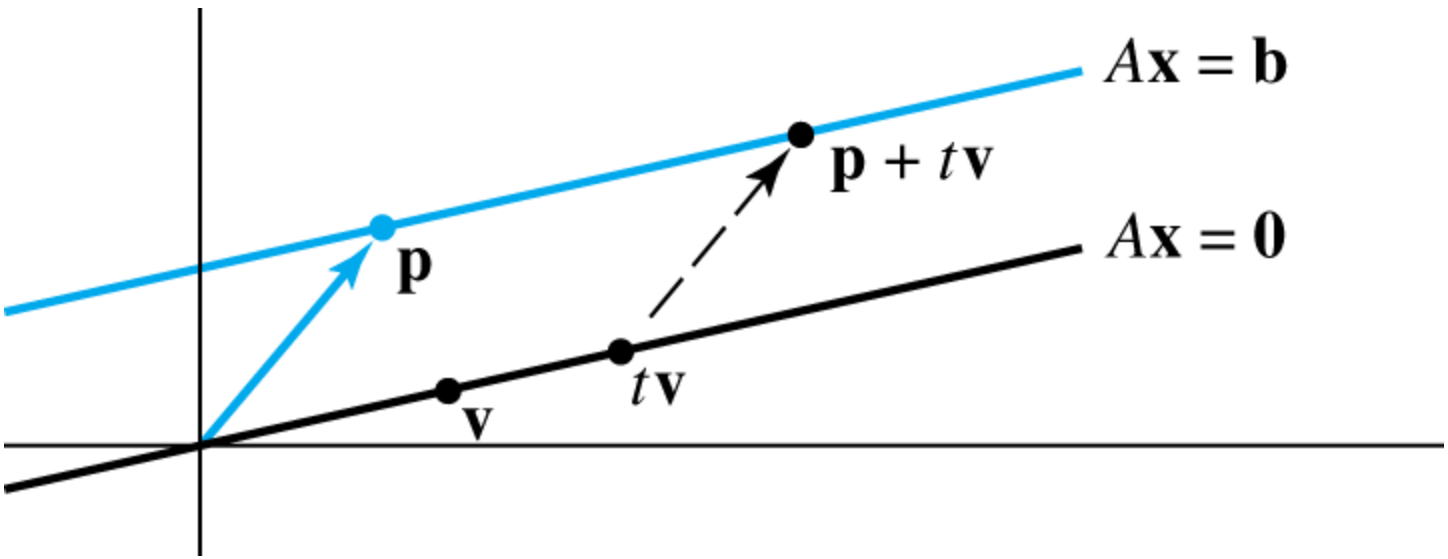
- If each point on a line L in $\mathbb{R}^2$ or $\mathbb{R}^3$ is translated by a vector **p**, the result is a line parallel to L. See the following figure.



SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- Suppose L is the line through **0** and **v**, described by equation (2).
- Adding **p** to each point on L produces the translated line described by equation (1).
- We call (1) **the equation of the line through p parallel to v**.
- Thus the solution set of $A\mathrm{x} = \mathrm{b}$ *is a line through* **p** *parallel to the solution set of* $A\mathrm{x} = 0$. The figure on the next slide illustrates this case.

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS



- The relation between the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ shown in the figure above generalizes to any consistent equation $A\mathbf{x} = \mathbf{b}$, although the solution set will be larger than a line when there are several free variables.

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- **Theorem 6:** Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given **b**, and let **p** be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

- This theorem says that if $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, using any particular solution **p** of $A\mathbf{x} = \mathbf{b}$ for the translation.

WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.