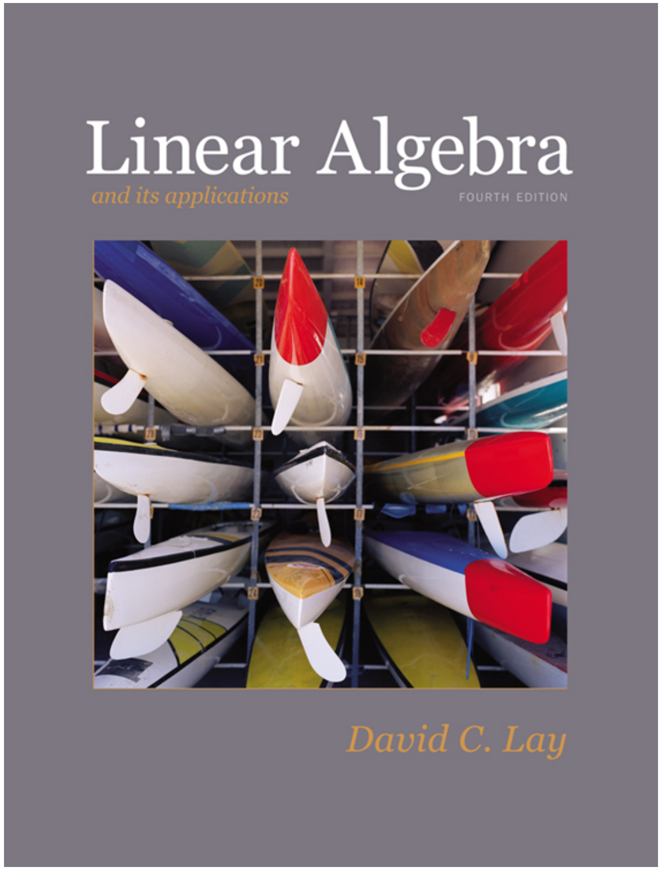
1 Linear Equations in Linear Algebra

1.3

VECTOR EQUATIONS



VECTOR EQUATIONS

Vectors in $\mathbb{R}^2$

- A matrix with only one column is called a **column vector**, or simply a **vector**.

- An example of a vector with two entries is

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where w_1 and w_2 are any real numbers.

- The set of all vectors with 2 entries is denoted by $\mathbb{R}^2$ (read "r-two").

VECTOR EQUATIONS

- The $\mathbb{R}$ stands for the real numbers that appear as entries in the vector, and the exponent 2 indicates that each vector contains 2 entries.
- Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal.
- Given two vectors **u** and **v** in $\mathbb{R}^2$, their **sum** is the vector $\mathbf{u} + \mathbf{v}$ obtained by adding corresponding entries of **u** and **v**.
- Given a vector **u** and a real number c, the **scalar multiple** of **u** by c is the vector $c\mathbf{u}$ obtained by multiplying each entry in **u** by c.

VECTOR EQUATIONS

- **Example 1:** Given $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4\mathbf{u}, (-3)\mathbf{v}$, and $4\mathbf{u} + (-3)\mathbf{v}$.

Solution: $4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$, $(-3)\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$ and

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

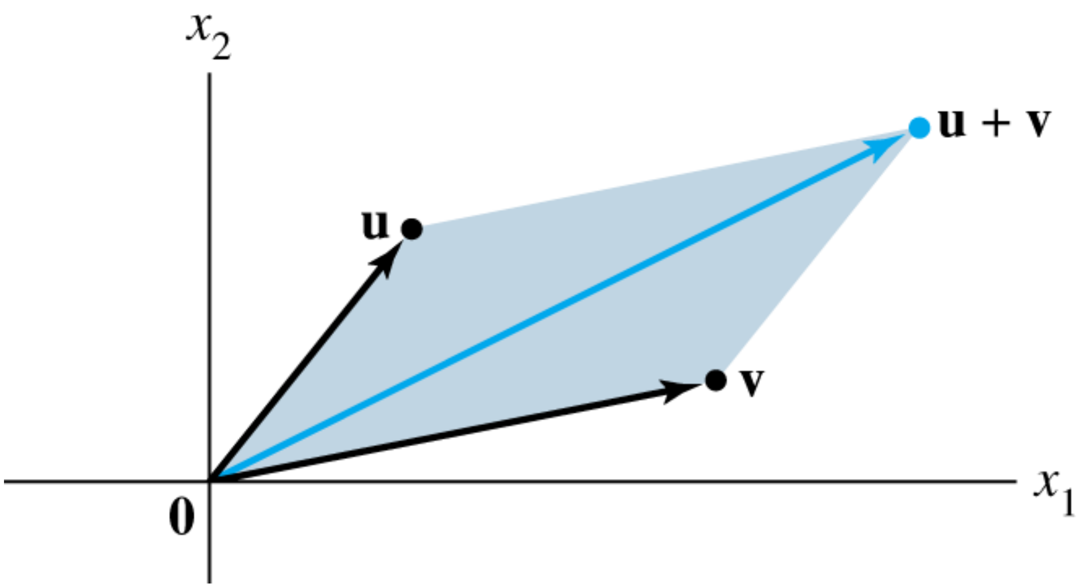
GEOMETRIC DESCRIPTIONS OF $\mathbb{R}^2$

- Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, *we can identify a geometric point (a, b) with the column vector* $\begin{bmatrix} a \\ b \end{bmatrix}$.

- So we may regard $\mathbb{R}^2$ as the set of all points in the plane.

PARALLELOGRAM RULE FOR ADDITION

- If **u** and **v** in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **u**, **0**, and **v**. See the figure below.



VECTORS IN $\mathbb{R}^3$ and $\mathbb{R}^n$

- Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries.
- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin.
- If n is a positive integer, $\mathbb{R}^n$ (read "r-n") denotes the collection of all lists (or *ordered n-tuples*) of n real numbers, usually written as $n\times1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$

- The vector whose entries are all zero is called the **zero vector** and is denoted by **0**.
- For all **u**, **v**, **w** in $\mathbb{R}^n$ and all scalars *c* and *d*:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,

where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

LINEAR COMBINATIONS

(vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$

(viii) $1\mathbf{u} = \mathbf{u}$

- Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$.

- The weights in a linear combination can be any real numbers, including zero.

LINEAR COMBINATIONS

- **Example 2:** Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

  Determine whether **b** can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \qquad \text{----(1)}$$

  If vector equation (1) has a solution, find it.

LINEAR COMBINATIONS

Solution: Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix},$$

$$\uparrow \mathbf{a}_1 \qquad \uparrow \mathbf{a}_2 \qquad \uparrow \mathbf{a}_3$$

which is same as

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

LINEAR COMBINATIONS

and $\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. ----(2)

- The vectors on the left and right sides of (2) are equal if and only if their corresponding entries are both equal. That is, x_1 and x_2 make the vector equation (1) true if and only if x_1 and x_2 satisfy the following system.

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \qquad \text{----(3)} \\ -5x_1 + 6x_2 &= -3 \end{aligned}$$

LINEAR COMBINATIONS

- To solve this system, row reduce the augmented matrix of the system as follows.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence **b** is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with weights $x_1 = 3$ and $x_2 = 2$. That is,

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

LINEAR COMBINATIONS

- Now, observe that the original vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$ are the columns of the augmented matrix that we row reduced:

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

$$\uparrow \quad \uparrow \quad \uparrow$$

$$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}$$

- Write this matrix in a way that identifies its columns.

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{bmatrix} \qquad \text{----(4)}$$

LINEAR COMBINATIONS

- A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

  has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}. \qquad \text{----(5)}$$

- In particular, **b** can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

LINEAR COMBINATIONS

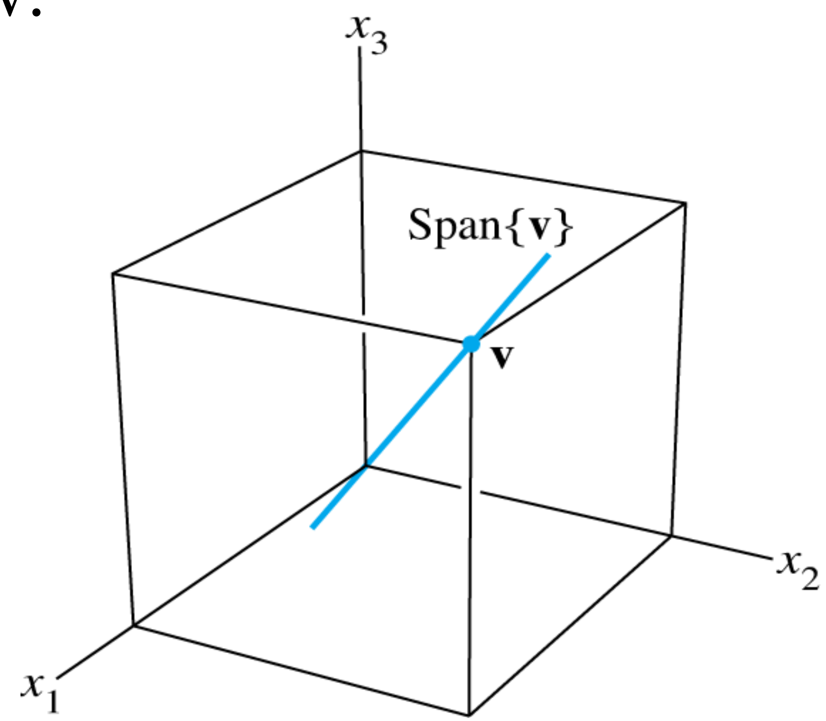
- **Definition:** If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by $\mathbf{v}_1, \ldots, \mathbf{v}_p$**. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \ldots + c_p\mathrm{v}_p$$

with $c_1, \ldots, c_p$ scalars.

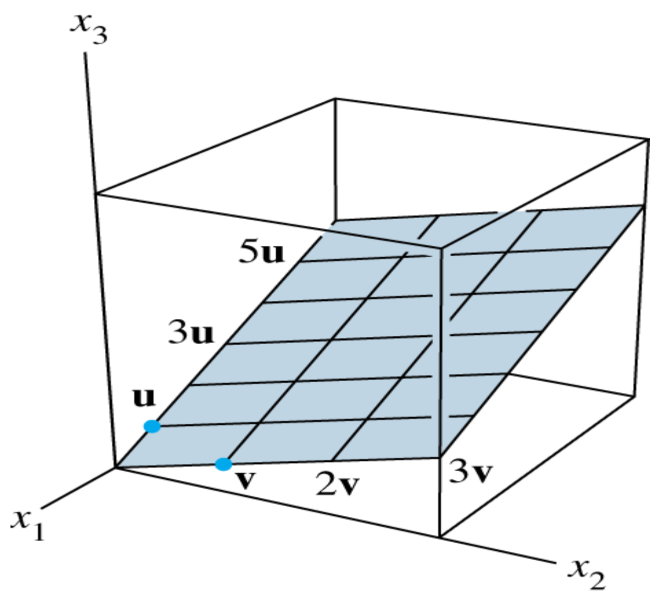
A GEOMETRIC DESCRIPTION OF SPAN {V}

- Let **v** be a nonzero vector in $\mathbb{R}^3$. Then Span $\{\mathbf{v}\}$ is the set of all scalar multiples of **v**, which is the set of points on the line in $\mathbb{R}^3$ through **v** and **0**. See the figure below.



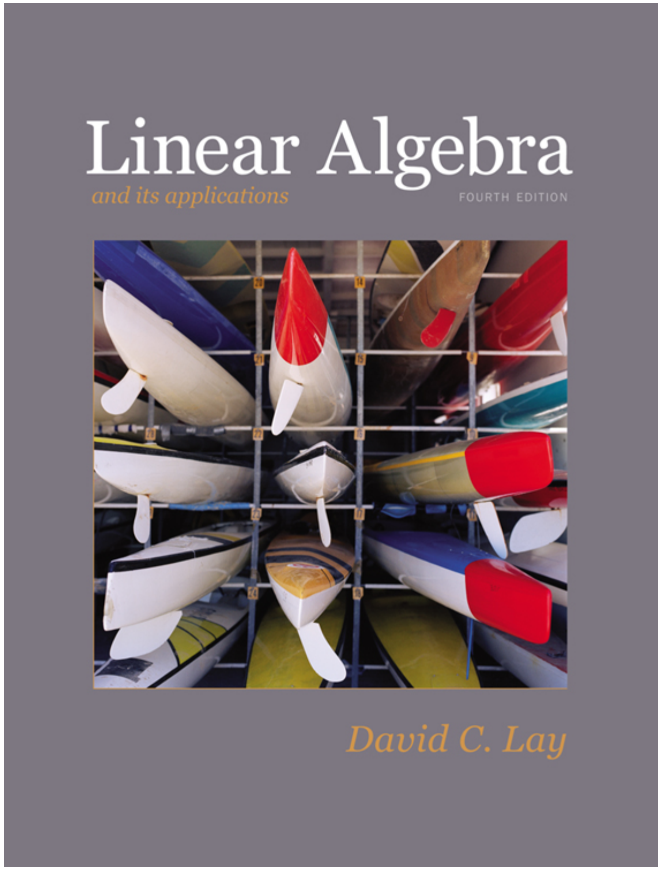
A GEOMETRIC DESCRIPTION OF SPAN {U, V}

- If **u** and **v** are nonzero vectors in $\mathbb{R}^3$, with **v** not a multiple of **u**, then Span {**u, v**} is the plane in $\mathbb{R}^3$ that contains **u**, **v**, and **0**.
- In particular, Span {**u**, **v**} contains the line in $\mathbb{R}^3$ through **u** and **0** and the line through **v** and **0**. See the figure below.



1 Linear Equations in Linear Algebra

1.4 THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$



MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Definition:** If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of** A **and x**, denoted by $A\mathbf{x}$, **is the linear combination of the columns of** A **using the corresponding entries in x as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

- $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Example 1:** For $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.
- **Solution:** Place $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ into the columns of a matrix A and place the weights $3, -5$, and 7 into a vector $\mathbf{x}$.
- That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x}$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- Now, write the system of linear equations as a vector equation involving a linear combination of vectors.
- For example, the following system

$$x_1 + 2x_2 - x_3 = 4 \quad \text{----(1)}$$
$$-5x_2 + 3x_3 = 1$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{----(2)}$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- As in the given example (1), the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \text{-----(3)}$$

- Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a **matrix equation**.

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Theorem 3:** If A is an $m \times n$ matrix, with columns a1, …, an, and if b is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n a_n = \mathbf{b},$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathrm{L} & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

EXISTENCE OF SOLUTIONS

- The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if **b** is a linear combination of the columns of A.
- **Theorem 4:** Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.
  a. For each **b** in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
  b. Each **b** in $\mathbb{R}^m$ is a linear combination of the columns of A.
  c. The columns of A span $\mathbb{R}^m$.
  d. A has a pivot position in every row.

PROOF OF THEOREM 4

- Statements (a), (b), and (c) are logically equivalent.
- So, it suffices to show (for an arbitrary matrix A) that (a) and (d) are either both true or false.
- Let U be an echelon form of A.
- Given **b** in $\mathbb{R}^m$, we can row reduce the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ to an augmented matrix $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ for some **d** in $\mathbb{R}^m$:

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \sim \ldots \sim \begin{bmatrix} U & \mathbf{d} \end{bmatrix}$$

- If statement (d) is true, then each row of U contains a pivot position, and there can be no pivot in the augmented column.

PROOF OF THEOREM 4

- So $A\mathbf{x} = \mathbf{b}$ has a solution for any **b**, and (a) is true.
- If (d) is false, then the last row of U is all zeros.
- Let **d** be any vector with a 1 in its last entry.
- Then $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ represents an *inconsistent* system.
- Since row operations are reversible, $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ can be transformed into the form $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$.
- The new system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, and (a) is false.

COMPUTATION OF $A\mathbf{x}$

- **Example 2:** Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.
- **Solution:** From the definition,

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

COMPUTATION OF $A\mathbf{x}$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \text{---}(1)$$

$$= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}.$$

- The first entry in the product $A\mathbf{x}$ is a sum of products (*a dot product*), using the first row of A and the entries in $\mathbf{x}$.

COMPUTATION OF $A\mathbf{x}$

- That is, $$\begin{bmatrix} 2 & 3 & 4 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ \\ \end{bmatrix}.$$

- Similarly, the second entry in $A\mathbf{x}$ can be calculated by multiplying the entries in the second row of A by the corresponding entries in $\mathbf{x}$ and then summing the resulting products.

$$\begin{bmatrix} & & \\ -1 & 5 & -3 \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ -x_1 + 5x_2 - 3x_3 \\ \\ \end{bmatrix}$$

ROW-VECTOR RULE FOR COMPUTING $A\mathbf{x}$

- Likewise, the third entry in $A\mathbf{x}$ can be calculated from the third row of A and the entries in $\mathbf{x}$.
- If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vertex $\mathbf{x}$.
- The matrix with 1s on the diagonal and 0s elsewhere is called an **identity matrix** and is denoted by I.
- For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT *A***x**

- **Theorem 5:** If A is an $m \times n$ matrix, u and v are vectors in $\mathbb{R}^n$, and c is a scalar, then
  a. $A(\mathrm{u}+\mathrm{v}) = A\mathrm{u} + A\mathrm{v}$;
  b. $A(c\mathrm{u}) = c(A\mathrm{u})$.
- **Proof:** For simplicity, take $n = 3$, $A = [\mathrm{a}_1 \quad \mathrm{a}_2 \quad \mathrm{a}_3]$, and **u**, **v** in $\mathbb{R}^3$
- For $i = 1, 2, 3$, let u_i and v_i be the ith entries in **u** and **v**, respectively.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

- To prove statement (a), compute $A(\mathbf{u}+\mathbf{v})$ as a linear combination of the columns of A using the entries in $\mathbf{u}+\mathbf{v}$ as weights.

$$A(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$$

Entries in $\mathbf{u}+\mathbf{v}$

$$= (u_1+v_1)\mathbf{a}_1 + (u_2+v_2)\mathbf{a}_2 + (u_3+v_3)\mathbf{a}_3$$

Columns of A

$$= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3)$$

$$= A\mathbf{u} + A\mathbf{v}$$

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

- To prove statement (b), compute $A(c\mathbf{u})$ as a linear combination of the columns of A using the entries in $c\mathbf{u}$ as weights.

$$A(c\mathbf{u}) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + (cu_3)\mathbf{a}_3$$

$$= c(u_1\mathbf{a}_1) + c(u_2\mathbf{a}_2) + c(u_3\mathbf{a}_3)$$

$$= c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3)$$

$$= c(A\mathbf{u})$$