01/25/2018

Recap: Vectors in $\mathbb{R}^n$ $\longleftarrow$ $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

Equality of two matrices

Addition: $\vec{u} + \vec{v}$

Scalar multiplication $c\vec{u}$

zero vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

A linear combination of $\vec{v}_1, \cdots, \vec{v}_p$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$$

Def: $\text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\} = \{\text{all linear combinations of } \vec{v}_1, \cdots, \vec{v}_p\}$

Statement: vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_p\vec{a}_p = \vec{b}$ is consistent

$\Longleftrightarrow$ $\vec{b}$ is a linear combination of $\vec{v}_1, \cdots, \vec{v}_p$

$\Longleftrightarrow$ $\vec{b}$ is in $\text{span}\{\vec{v}_1, \cdots, \vec{v}_p\}$

Matrix-vector multiplication $A\vec{x}$

A: $m \times n$ matrix

$A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$, $\vec{a}_j$ is in $\mathbb{R}^m$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\vec{x}$ is in $\mathbb{R}^n$.

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

$A\vec{x}$ is in $\mathbb{R}^m$

Caution: $\underbrace{A}_{m\times n}\ \underbrace{\vec{x}}_{n\times 1}$ is defined

only when # of columns of A

= # of entries of $\vec{x}$

Theorem 3:

Consider matrix $A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$

Matrix equation $A\vec{x} = \vec{b}$ is equivalent to

vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_n\vec{a}_n = \vec{b}$

which is equivalent to the linear system

whose augmented matrix is

$$[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n \mid \vec{b}]$$

Q: Is $A\vec{x}=\vec{b}$ consistent for all $\vec{b}$?

EX. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ -4 & 6 & -2 & b_2 \\ -3 & 1 & -4 & b_3 \end{array}\right]$

$\xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & b_1 \\ 0 & \boxed{14} & 10 & b_2+4b_1 \\ 0 & 0 & 0 & \underline{b_3+b_1-\frac{1}{2}b_2} \end{array}\right]$

When $b_3+b_1-\frac{1}{2}b_2 \neq 0 \Rightarrow$ inconsistent.

Observation:

Row 3 of matrix A does not have a pivot position

$\Rightarrow$ When $b_3+b_1-\frac{1}{2}b_2 \neq 0$ it is a pivot position

$\Rightarrow$ The rightmost column of $[A \mid \vec{b}]$ is a pivot column.

$\Rightarrow$ $A\vec{x}=\vec{b}$ is not consistent for all $\vec{b}$.

Ex.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ -4 & 6 & -2 & b_2 \\ -3 & 1 & -1 & b_3 \end{array}\right]$$

$$\xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & b_1 \\ 0 & \boxed{14} & 10 & b_2+4b_1 \\ 0 & 0 & \boxed{3} & b_3+b_1-\frac{1}{2}b_2 \end{array}\right]$$

Observation:

Every row of matrix A has a pivot position.

$\Rightarrow$ All pivot positions of $[A \mid \vec{b}]$ are in matrix A.

$\Rightarrow$ Right most column of $[A \mid \vec{b}]$ is not a pivot column.

$\Rightarrow$ $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$.

Theorem 4: Consider an $m \times n$ matrix $A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$

4 statements below are equivalent

a. $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$

b. Every $\vec{b}$ in $\mathbb{R}^m$ is in span$\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$.

c. Span$\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\} = \mathbb{R}^m$

d. Matrix A has a pivot position in every row.

Row-vector rule for computing $A\vec{x}$

The dot product of a row and a vector

$$\begin{bmatrix} a_1 & a_2 \cdots a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1x_1 + a_2x_2 + a_3x_3 \end{bmatrix}$$

Let us do

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \hline a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}.$$

Row-vector rule:

The i-th entry of $A\vec{x}$ is the doc product of the i-th row ~~and~~ of A and $\vec{x}$.

Ex. $\begin{bmatrix} 2 & 1 & 3 \\ 6 & -2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\times4+1\times2+3\times3 \\ 6\times4-2\times2+4\times3 \end{bmatrix} = \begin{bmatrix} 19 \\ 32 \end{bmatrix}$

Identity matrix

2×2 identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

It satisfies $I\vec{x} = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^2$

3×3 identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

It satisfies $I\vec{x} = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^3$

$n\times n$ identity matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

It satisfies $I\vec{x} = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$.

properties of $A\vec{x}$

*) $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

*) $A(c\vec{u}) = cA\vec{u}$

*). $I\vec{x} = \vec{x}$

*) $A\vec{0} = \vec{0}$

Sec 1.5 Structure of the solution set

Homogeneous linear system

$$A\vec{x} = \vec{0}$$

It is always consistent. $A\vec{0} = \vec{0}$

$\vec{x} = \vec{0}$ is called the trivial solution.

Q: Is there a non-trivial solution?

Augmented matrix

$[A \mid 0]$

Row reduction to an echelon form

Identify basic variables and free variables.

If there is no free variable $\Rightarrow$ $\vec{x} = 0$ is the only solution.

If there is at least one free variable

$\Rightarrow$ it has a non-trivial solution.

Statement: $A\vec{x} = \vec{0}$ has a non-trivial solution.
if and only if it has at least
one free variable.

Parametric form of the solution set

Ex. $2x_1 - 3x_2 + 5x_3 = 0$

Augmented matrix

$$\left[\begin{array}{ccc|c} \boxed{2} & -3 & 5 & 0 \end{array}\right]$$

Multiply by $\frac{1}{2}$

$$\left[\begin{array}{ccc|c} \boxed{1} & -\frac{3}{2} & \frac{5}{2} & 0 \end{array}\right]$$ reduced echelon form

Basic variable: x_1

Free variables: x_2, x_3

$$x_1 = \frac{3}{2}x_2 - \frac{5}{2}x_3 .$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x_2 - \frac{5}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}$$

parametric form.

In the notation of span{ }

The solution set = Span$\left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

EX. $\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -6 & -7 & 9 & 0 \\ 6 & 1 & -11 & 0 \end{array}\right]$ Augmented matrix of the linear system.

$\xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -9 & -3 & 0 \end{array}\right]$

$\longrightarrow \left[\begin{array}{ccc|c} \boxed{3} & 5 & -4 & 0 \\ 0 & \boxed{3} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Basic variables: x_1, x_2

Free variable: x_3

To reduced echelon form

$\left[\begin{array}{ccc|c} 1 & 0 & -17/9 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17/9\, x_3 \\ -\frac{1}{3} x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 17/9 \\ -1/3 \\ 1 \end{bmatrix}$$

parametric form of the solution set.

$$\text{The solution set} = \operatorname{Span}\left\{ \begin{bmatrix} 17/9 \\ -1/3 \\ 1 \end{bmatrix} \right\}$$

<u>Non-homogeneous system</u>

$$A\vec{x} = \vec{b}$$

<u>Ex</u>. Augmented matrix of the linear system

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 2 \\ -6 & -7 & 9 & -1 \\ 6 & 1 & -11 & -5 \end{array}\right]$$

$$\xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -17/9 & -1 \\ 0 & \boxed{1} & 1/3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Longrightarrow$ It is consistent.

Basic variables: x_1, x_2

Free variable: x_3

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + 17/9\, x_3 \\ 1 - 1/3\, x_3 \\ x_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\text{A particular solution of } A\vec{x}=\vec{b}} + \underbrace{x_3 \begin{bmatrix} 17/9 \\ -1/3 \\ 1 \end{bmatrix}}_{\text{The solution set of } A\vec{x}=\vec{0}}$$

The solution set

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \text{Span}\left\{ \begin{bmatrix} 17/9 \\ -1/3 \\ 1 \end{bmatrix} \right\}$$

<u>Theorem 6</u>: Suppose $A\vec{x} = \vec{b}$ has a solution. $\vec{p}$

The solution set of $A\vec{x} = \vec{b}$

$= \vec{p}$ + the solution set of $A\vec{x} = \vec{0}$

<u>Sec 1.7</u>: Linear independence

<u>Def</u>: $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is said to be linearly independent if the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution.

$\{\vec{v}_1, \cdots, \vec{v}_p\}$ is said to be linearly dependent if

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

has a non-trivial solution.

Linear dependence

$\Rightarrow$ There exists $c_1, c_2, \cdots, c_p$, not all zeros. such that

$$\underline{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}}$$

This is called a linear dependence relation. among $\vec{v}_1, \cdots, \vec{v}_p$.

Consider matrix $A = [\,\vec{a}_1 \;\; \vec{a}_2 \;\cdots\; \vec{a}_n\,]$

$m \times n$ matrix

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$$

is equivalent to

$$A\vec{x} = \vec{0}$$

Statement: The columns of matrix A are linearly dependent if and only if $A\vec{x} = \vec{0}$ has a non-trivial solution.