01/23/2018

Recap

Echelon form: properties 1)
2)
3)

Reduced echelon form: properties 4)
5)

Theorem 1: A matrix is row equivalent to one and only one reduced echelon matrix.

Row reduction algorithm:

Forward phase: Steps 1
2
3

Backward phase:

Definition: pivot position
pivot column

Solving a linear system using row reduction algorithm

*) Row reduce the augmented matrix
$\longrightarrow$ reduced echelon form

*) Basic variables

*) Free variables

*) General solution

Theorem 2 (existence and uniqueness)

*) A linear system is consistent if and only if the right most column of its augmented matrix is not a pivot column.

*) If it is consistent and there is no free variable, then the solution is unique.

*) If it is consistent and there is at least one free variable, then

Solution set = infinitely many solutions.

Sec. 1.3 Vector equations

A column vector of n entries ~~is a $n\times 1$~~ is an $n \times 1$ matrix

Ex. $\vec{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is a vector of 2 entries.

Equality of two matrices:

Two matrices are equal if and only if

*) they have the same size.

*) The corresponding entries are the same.

Ex $\begin{bmatrix} 2 \\ -3 \end{bmatrix} \neq \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 2 \\ -3 \end{bmatrix} \neq \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$

Vectors in $\mathbb{R}^2$:

$\mathbb{R}^2$ = the set of all vectors with 2 entries.

"$\vec{u}$ is in $\mathbb{R}^2$" $\Longleftrightarrow$ $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Addition:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Scalar multiplication

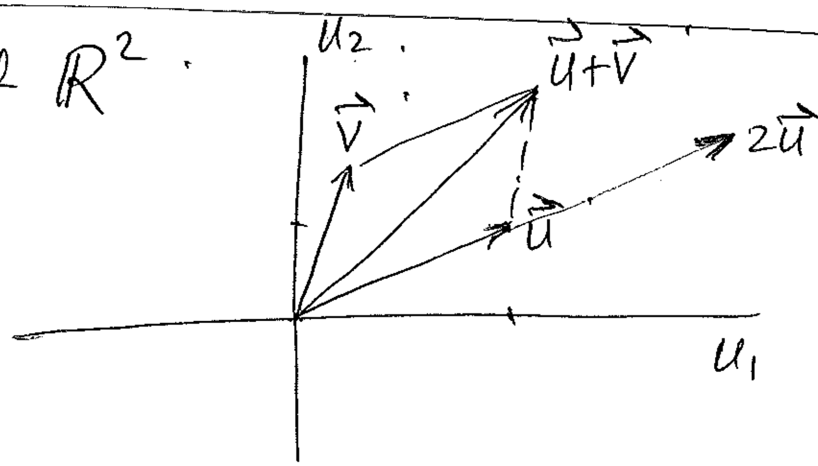
$$c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$$

Ex. $\vec{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$$2\vec{u} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}, \qquad (-3)\vec{v} = \begin{bmatrix} -3 \\ 12 \end{bmatrix}$$

$$2\vec{u} + (-3)\vec{v} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

Geometry of $\mathbb{R}^2$.



Vectors in $\mathbb{R}^3$

$\mathbb{R}^3$ = the set of all vectors with <u>3 entries</u>

Vectors in $\mathbb{R}^n$

$\mathbb{R}^n$ = the set of all vectors with <u>n entries</u>.

"$\vec{u}$ is in $\mathbb{R}^n$" $\Longleftrightarrow$ $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

<u>Addition:</u>

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

<u>Scalar multiplication</u>

$$c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$

$$-\vec{v} = (-1)\vec{v}$$

$$\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$$

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

properties of $\vec{u}+\vec{v}$ and $c\vec{u}$

1) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

2). $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

We can write $\vec{u}+\vec{v}+\vec{w}$

3). $\vec{u}+\vec{0}=\vec{u}$

4). $\vec{u}-\vec{u}=\vec{0}$

5). $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$

6). $(c+d)\vec{u} = c\vec{u}+d\vec{u}$

7). $c(d\vec{u}) = (cd)\vec{u}$

8). $1\vec{u}=\vec{u}$

<u>Def</u>. Linear combination of vectors.

vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p$ — $\vec{v}_j$ is in $\mathbb{R}^n$

Scalars: $c_1, c_2, \cdots, c_p$.

$$c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_p\vec{v}_p$$

is called a linear combination of $\vec{v}_1, \cdots, \vec{v}_p$

with weights $c_1, c_2, \cdots, c_p$.

EX. $\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

Q: Is $\vec{b}$ a linear combination of $\vec{a}_1$ and $\vec{a}_2$?

If so find the weights.

Consider the vector equations.

$$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$$

$\vec{b}$ is a linear combination of $\vec{a}_1$ and $\vec{a}_2$

if and only if the vector equation has a solution.

$$x_1\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 3x_2 &= 7 \\ -2x_1 + 3x_2 &= 4 \\ -5x_1 + x_2 &= -3 \end{aligned}$$

Augmented matrix

$$\begin{array}{c} \left[\begin{array}{cc|c} 1 & 3 & 7 \\ -2 & 3 & 4 \\ -5 & 1 & -3 \end{array}\right] \\ \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{b} \end{array} \end{array} \longrightarrow [\vec{a}_1 \ \vec{a}_2 \mid \vec{b}]$$

We can go directly from

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$$

to $[\vec{a}_1 \ \vec{a}_2 \mid \vec{b}]$

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ -2 & 3 & 4 \\ -5 & 1 & -3 \end{array}\right]$$

Add $2 \times (\text{row } 1)$ to (row 2)

Add $5 \times R1$ to $R3$

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array}\right]$$

Add $(-16/9) \times R2$ to $R3$.

$$\left[\begin{array}{cc|c} \boxed{1} & 3 & 7 \\ 0 & \boxed{9} & 18 \\ 0 & 0 & 0 \end{array}\right]$$

echelon form

Multiply $R2$ by $1/9$

Add $(-3) \times R2$ to $R1$

$$\begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

reduced echelon form

Basic variables : x_1, x_2

Free variable : none

$$x_2 = 2$$
$$x_1 = 1.$$

Solution: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Yes: $\vec{b} = \vec{a}_1 + 2\vec{a}_2$

Statement*

Vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_p\vec{a}_p = \vec{b}$ is equivalent to the linear system whose augmented matrix is $[\vec{a}_1\ \vec{a}_2\ \cdots \vec{a}_p \mid \vec{b}]$

Def: $\text{span}\{\vec{v}_1, \cdots, \vec{v}_p\} = \{$all linear combinations of $\vec{v}_1, \vec{v}_2, \cdots \vec{v}_p\}$.

The subset spanned by $\{\vec{v}_1, \cdots, \vec{v}_p\}$.

$\vec{b}$ is a linear combination of $\{\vec{v}_1, \cdots, \vec{v}_p\}$.

$\iff$ $\vec{b}$ is in $\text{span}\{\vec{v}_1, \cdots, \vec{v}_p\}$.

Geometry description of span{ }.

*). If $\vec{v} \neq \vec{0}$, then

$\text{span}\{\vec{v}\}$ = the line going through $\vec{v}$ and $\vec{0}$.

*). If $\vec{u} \neq \vec{0}$, $\vec{v} \neq \vec{0}$ and $\vec{u}$ is not a multiple of $\vec{v}$, then

$\text{Span}\{\vec{u}, \vec{v}\}$ = the plane going through $\vec{u}$, $\vec{v}$ and $\vec{0}$

Sec 1.4. matrix equation $A\vec{x} = \vec{b}$

Def. $A\vec{x}$.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$ $m \times n$ matrix

$$A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n].$$

$$\vec{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$ $\boxed{\text{is in } \mathbb{R}^m}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ $\boxed{\text{is in } \mathbb{R}^n}$

$A\vec{x}$ is ~~a vecto~~ a vector of m entries.

$A\vec{x}$ is in $\mathbb{R}^m$

$$\boxed{A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n}$$

Caution $A\vec{x}$ makes sense only if

\# of columns of A = \# of entries ~~of x~~ of $\vec{x}$

$$\underset{m\times n}{A}\ \ \underset{n\times 1}{\vec{x}}$$

Ex. $\begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 29 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ = <u>not defined</u>

Theorem 3

Consider matrix $A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$

Matrix equation $A\vec{x} = \vec{b}$ is equivalent to

vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$

which is equivalent to the linear system

whose augmented matrix is

$$[\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n \mid \vec{b}]$$

Statement :

$A\vec{x} = \vec{b}$ is consistent if and only if

$\vec{b}$ is in span$\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$.

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 6 & -2 \\ -3 & 1 & -4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Q: Is $A\vec{x} = \vec{b}$ consistent for all $\vec{b}$ in $\mathbb{R}^3$?

A: We use row reduction + Theorem 2

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ -4 & 6 & -2 & b_2 \\ -3 & 1 & -4 & b_3 \end{array}\right]$$

Forward phase.

Row 1

Add $4 \times R1$ to $R2$.
Add $3 \times R1$ to $R3$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{array}\right]$$

Row 2

Add $\left(-\frac{1}{2}\right) \times R2$ to $R3$.

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & b_1 \\ 0 & \boxed{14} & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + b_1 - \frac{1}{2}b_2 \end{array}\right]$$

If $b_3 + b_1 - \frac{1}{2}b_2 \neq 0$, it is a pivot position.

Answer: No. $A\vec{x} = \vec{b}$ is not consistent for all $\vec{b}$ in $\mathbb{R}^3$.