01/18/2018

Recap :

Linear system

Solution set

Equivalence

3 possibilities

no solution

exactly one solution

infinitely many solutions

Consistency

Matrix notation

coefficient matrix

augmented matrix

Strategy for solving a linear system

Transform it to a new system

*) that has the triangular form

*) and that is equivalent to the original system.

Elementary Row Operations (EROs)

*) Add a multiple of row j to row K, $j \neq K$

*) Interchange two rows

*) Multiply a row by a non-zero constant

All EROs are reversible.

The linear systems before and after EROs are equivalent.

Row equivalence of two matrices.

Augmented matrices are row equivalent
$\Longrightarrow$ Linear systems are equivalent.

Row reduction and echelon forms

Terminology

*) a non-zero row

*) a row of all zeros

*) the leading entry of a non-zero row.

*) "row reduce" = "transform using EROs"

Def. Echelon form and reduced echelon form

We say a matrix is in echelon form if it satisfies properties 1, 2 and 3.

1. All non-zero rows are above rows of all zeros.
2. The leading entry of each non-zero row is in a column to the right of the leading entry of the row above it.
3. All entries below a leading entry are zeros.

We say it is in reduced echelon form ~~if~~ if it also satisfies properties 4 and 5

4. Each leading entry is 1
5. Each leading entry is the only non-zero in its column.

EX.

$$\begin{pmatrix} 4 & 7 & 2 & 3 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

it is in echelon form

not reduced echelon form

$$\begin{pmatrix} 1 & 7 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

it is in reduced echelon form

$$\begin{pmatrix} 4 & 7 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 2 \end{pmatrix}$$ not echelon form

$$\begin{pmatrix} 0 & 0 & 5 & 2 \\ 4 & 7 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ not echelon form

Echelon matrix

Reduced echelon matrix

Theorem 1: A matrix is row equivalent to one and only one reduced echelon matrix

The row reduction algorithm:

Forward phase:

Goal: row reduce the matrix to an echelon form.

Start with row 1, then move onto row 2, ...

For row k, we have 3 steps.

Step 1: Look at the submatrix consisting of row k and all rows below it.

Find the left most non-zero column of the submatrix ← pivot column

Step 2: look at the intersection of row K and the pivot column. ← pivot position

We need a non-zero entry at the pivot position (interchange two rows if necessary)

Step 3: Use EROs to eliminate entries below the pivot position.

Repeat Steps 1, 2 and 3 for next row.

Summary of the forward phase

Step 1. identify the pivot column

Step 2. make the entry at the pivot position non-zero.

Step 3. Use ERO's to create zeros below the pivot position.

Backward phase:

Goal: row reduce it to the reduced echelon form.

For each non-zero row starting from bottom,

*) Make the leading entry 1 by scaling.
Create zeros above the leading entry using EROs

Def: pivot position = position of a leading entry in echelon form

pivot column = a column containing a pivot position.

Ex
$$\begin{pmatrix} 0 & 0 & -3 & -6 & 4 & 9 \\ -1 & -1 & -2 & -1 & 3 & 1 \\ -2 & -2 & -3 & 0 & 3 & -1 \\ 1 & 1 & 4 & 5 & -9 & -7 \end{pmatrix}$$

Forward phase.

$\boxed{\text{Row 1}}$. Step 1. identify pivot column

Step 2. make the pivot non-zero

$$\begin{pmatrix} 1 & 1 & 4 & 5 & -9 & -7 \\ \boxed{-1} & -1 & -2 & -1 & 3 & 1 \\ \boxed{-2} & -2 & -3 & 0 & 3 & -1 \\ 0 & 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

$\boxed{\text{Step 3}}$ Add row 1 to row 2

Add 2 x (row 1) to row 3

$$\begin{pmatrix} 1 & 1 & 4 & 5 & -9 & -7 \\ 0 & 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & \boxed{5} & 10 & -15 & -15 \\ 0 & 0 & \boxed{-3} & -6 & 4 & 9 \end{pmatrix}$$

$\boxed{\text{Row 2}}$

<u>Step 1</u>:

<u>Step 2</u>:

<u>Step 3</u>:

Add $\left(-\frac{5}{2}\right) \times$ row 2 to row 3

Add $\left(\frac{3}{2}\right) \times$ row 2 to row 4

$$\begin{pmatrix} 1 & 1 & 4 & 5 & -9 & -7 \\ 0 & 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 \end{pmatrix}$$

$\boxed{\text{Row 3}}$ <u>Step 1</u>, Step 2

$$\begin{pmatrix} \boxed{1} & 1 & 4 & 5 & -9 & -7 \\ 0 & 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & \boxed{-5} & 0 \\ 0 & 0 & 0 & 0 & \boxed{0} & 0 \end{pmatrix}$$

Echelon form

$\boxed{\text{Row 4}}$: $\boxed{\text{done with the forward phase}}$

<u>Backward phase</u>:

$\boxed{\text{Row 3}}$ Multiply row 3 by $\frac{-1}{5}$

$$\begin{pmatrix} 1 & 1 & 4 & 5 & 0 & -7 \\ 0 & 0 & \boxed{2} & 4 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Add $6 \times$(row 3) to (row 2)
Add $9 \times$(row 3) to (row 1)

$\boxed{\text{Row 2}}$ Multiply (row 2) by $\frac{1}{2}$

$$\left(\begin{array}{ccccc|c} \boxed{1} & 1 & 0 & -3 & 0 & 5 \\ 0 & 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

← Reduced echelon form

Add $(-4) \times$(row 2) to (row 1)

$\boxed{\text{Row 1}}$ Done with the backward phase.

Solving a linear system using row reduction algorithm.

Ex.
$$\left(\begin{array}{ccccc|c} 0 & 0 & -3 & -6 & 4 & 9 \\ -1 & -1 & -2 & -1 & 3 & 1 \\ -2 & -2 & -3 & 0 & 3 & -1 \\ 1 & 1 & 4 & 5 & -9 & -7 \end{array}\right)$$

Row reduction algorithm.

Def: Basic variables and free variables.

"basic variables" = variables corresponding to pivot positions.

"free variables" = rest of variables.

Back to example

basic variables : x_1, x_3, x_5

free variables : x_2, x_4

Move all free variables to the RHS

Express basic variables in terms of free variables.

row 1. $x_1 + x_2 \quad -3x_4 = 5$

$$\Rightarrow \quad x_1 = 5 - x_2 + 3x_4.$$

row 2. $x_3 + 2x_4 = -3$

$$\Rightarrow \quad x_3 = -3 - 2x_4.$$

row 3. $x_5 = 0$.

$$\begin{cases} x_1 = 5 - x_2 + 3x_4. \\ x_2 = \text{free}. \\ x_3 = -3 - 2x_4 \\ x_4 = \text{free} \\ x_5 = 0 \end{cases}$$

solution set, general solution.

Q: What happens if the last column of the augmented matrix contains a pivot position.

Ex.

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

Row3:

$$0 = 1$$

$\Longrightarrow$ no solution, inconsistent.

Theorem 2. (existence and uniqueness).

*) A linear system is consistent if and only if the right most column of its augmented matrix is not a pivot column.

*). If it is consistent and there is no free variables, the the solution is unique.
(Solution set = exactly one solution)

*). If it is consistent and there is at least one free variable, then solution set = infinitely many solutions.

Ex. Augmented matrix

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 4 \end{array}\right)$$

Right most column is not a pivot column.

$\Longrightarrow$ it is consistent.

Basic variables: X_1, X_2, X_3.

No free variable.

Row 3: $X_3 = 4$.

Row 2: $X_2 = 2$

$\boxed{\text{Row 1}}$: $X_1 = 3$

$$\text{Solution set} = \begin{cases} X_1 = 3 \\ X_2 = 2 \\ X_3 = 4 \end{cases}$$

Ex. Augmented matrix

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 4 & 3 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Right most column is not a pivot column

$\Longrightarrow$ Consistency.

Basic variables: x_1, x_2

Free variable: x_3.

Row 2: $x_2 = 2 - x_3$

Row 1: $x_1 = 3 - 4x_3$.

$$\begin{cases} x_1 = 3 - 4x_3 \\ x_2 = 2 - x_3 \\ x_3 = \text{free}. \end{cases}$$

Solution set