01/16/2018

Recap

Euler formula

$$e^{\theta i} = \cos\theta + \sin\theta\, i$$

Exponential form

$$z = \underbrace{r}_{|z|}\, e^{\overbrace{\theta}^{\arg(z)} i}$$

Multiplication:

$$r_1 e^{\theta_1 i}\, r_2 e^{\theta_2 i} = r_1 r_2 e^{(\theta_1+\theta_2) i}$$

Division:

$$\frac{r_1 e^{\theta_1 i}}{r_2 e^{\theta_2 i}} = \left(\frac{r_1}{r_2}\right) e^{(\theta_1-\theta_2) i}$$

Factoring a complex polynomial

$$p(z) = c_n (z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_n)$$

Factoring a real polynomial

$$p(z) = a_n (z-\xi_1)\cdots(z-\xi_m)\, Q_1(z)\cdots Q_K(z)$$

where $Q_j(z)$ = quadratic form corresponding to a conjugate pair of roots.

n-th roots of a complex number u

*) Write u in the exponential form

$$u = re^{\theta i}$$

*) Solve $z^n = u$

let $z = pe^{wi}$

$$\Rightarrow (pe^{wi})^n = re^{\theta i}$$

$$\Rightarrow p^n e^{nwi} = re^{\theta i}$$

$$\Rightarrow p^n = r \Rightarrow p = \sqrt[n]{r}$$

$$nw = \theta + 2k\pi,$$

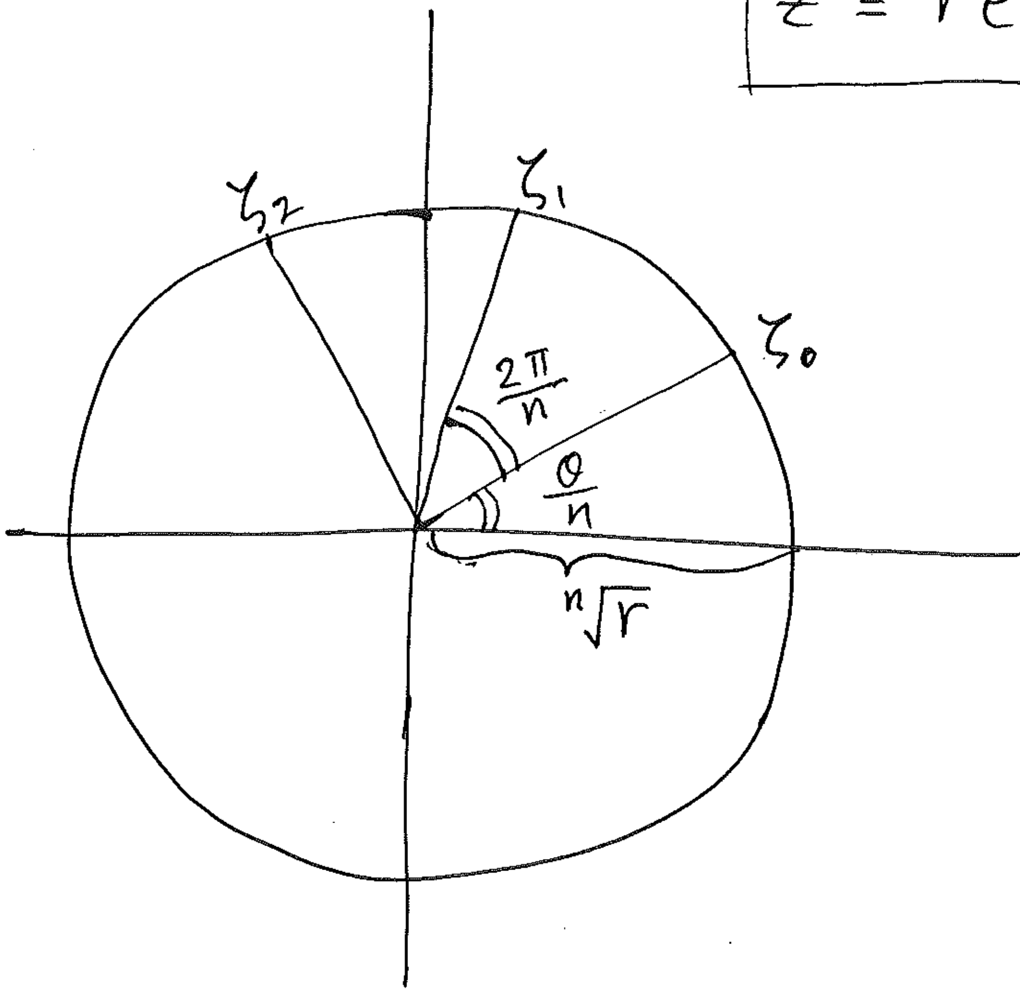
$$\Rightarrow w = \frac{\theta + 2k\pi}{n}, \quad k = 0, 1, 2, \cdots, (n-1)$$

<u>Summary</u>: $z^n = re^{\theta i}$ has n solutions

$$\zeta_k = \sqrt[n]{r}\, e^{\frac{\theta + 2k\pi}{n} i}, \quad k = 0, 1, 2, \cdots, (n-1)$$

Geometry of the n-th roots

$$\boxed{z^n = re^{\theta i}}$$



Section 1.1 Linear systems

Def. A linear equation in variables $x_1, x_2, \cdots, x_N$ is of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_N x_N = b$$

$a_1, a_2, \cdots, a_N, b$ are known

$x_1, x_2, \cdots, x_N$ are unknown

Def. A linear system

= a collection of linear eqs

Ex. $3x_1 - 5x_2 = 7. \rightarrow$ a linear eq.

$$(\text{Eq}1) \begin{cases} x_1 - 2x_2 = -1 \\ x_1 - x_2 = 1 \end{cases}$$ is a linear system

Def. A solution is a set of values for $(x_1, x_3 \cdots, x_N)$ such that the system is satisfied.

$(x_1, x_2) = (3, 2)$ is a solution of (Eq 1)

$3 - 2 \times 2 = -1$ ✓

$3 - 2 = 1$ ✓

Two things.

*) A linear sys may have no solution.

Ex. $\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = 3 \end{cases}$ has no solution.

*) A linear sys may have more than 1 solutions.

Ex $\begin{cases} x_1 - 2x_2 = 2 \\ 2x_1 - 4x_2 = 4 \end{cases}$

$(2, 0)$ is solution.

$(4, 1)$ is also a solution.

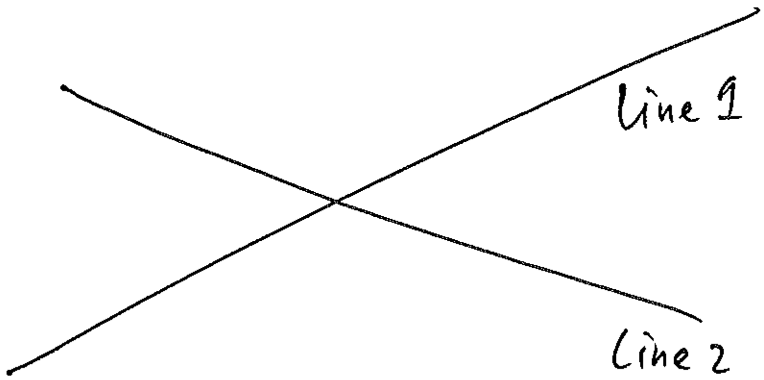
Def: Solution set = collection of all solutions.

Def: Two systems are equivalent to each other if their solution sets are the same.

Geometry of 2×2 systems.

$a_{11} x_1 + a_{12} x_2 = b_1$ — a straight line, Line 1

$a_{21} x_1 + a_{22} x_2 = b_2$ ← a straight line. Line 2

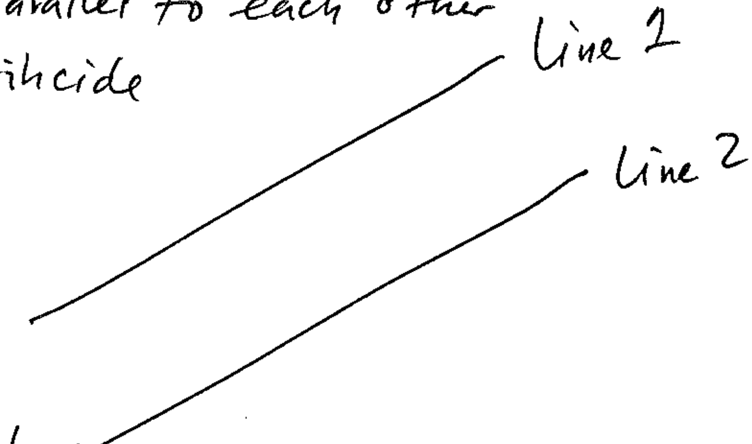


Three possibilities

*) Two lines are not parallel to each other.

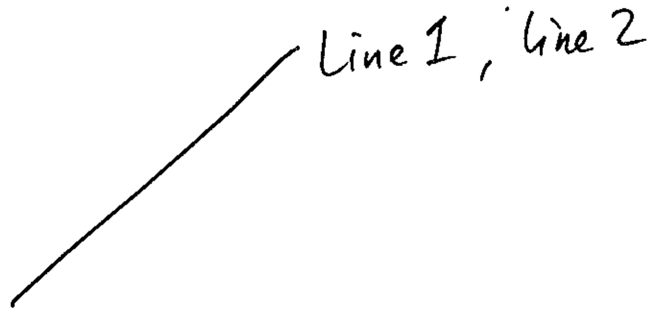
→ exact one solution

*) Two lines are parallel to each other but do not coincide



→ no solution.

*) Two lines coincide with each other.



→ infinitely many solutions

Def. consistent
in consistent.

Matrix notation

Def. an $m \times n$ matrix is a rectangular array of numbers of m rows and n columns.

Ex. $\begin{pmatrix} 2 & 4 & 1 \\ 3 & 7 & -5 \end{pmatrix}$ is a 2×3 matrix.

Ex. Consider.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 2x_2 - 6x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Coefficient matrix $\begin{pmatrix} 1 & -2 & 1 \\ 2 & -2 & -6 \\ -4 & 5 & 9 \end{pmatrix}$

Augmented matrix $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -2 & -6 & 8 \\ -4 & 5 & 9 & 9 \end{array}\right)$

Q. How to solve a linear system.

A special case.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

row 3: $x_3 = 3$.

row 2: $x_2 = 4 + 4x_3 = 16$

row 1: $x_1 = 2x_2 - x_3 = 32 - 3 = 29$

"Triangular form"

Strategy for solving a linear system.

*) Transform the original system to triangular form

*) The system after transformation should be equivalent to the original system.

Elementary row operations (EROs).

1. (Replacement) Add a multiple of row j to row k. $j \neq k$

2 Interchange two rows.

3 (Scaling) Multiply a row by a non-zero constant.

All EROs are reversible.!

Ex.

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -2 & -6 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right)$$

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - 2x_2 - 6x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

Add $(-2) \times$ row 1 to row 2

Add $4 \times$ row 1 to row 3.

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right)$$

Add $3 \times$ row 2 to row 3

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & \cancel{29}\ 3 \end{array}\right) \rightarrow \text{Solution} = \cancel{(3, 16, 29)}\ (29, 16, 3)$$

Q: Is the system after transform equivalent to the original system?

*) A solution of the original system is also a solution of the new system.

*) All EROs are reversible!

⇒ The two systems before and after EROs are equivalent.

Def. Row equivalence of matrices.

Two matrices are called row equivalent if one can be transformed to the other by EROs

Ex. Determine the consistency.

$$\left(\begin{array}{ccc|c} 1 & 5 & 2 & 6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 10 \end{array}\right)$$

row 3: $5X_3 = 10$ → solve for X_3.

row 2: $4X_2 = 2 + 7X_3$

row 1: $X_1 = -5X_2 - 2X_3 + 6$ → Consistent → Exact one solution.

Ex. determine consistency.

$$\left(\begin{array}{ccc|c} 1 & 5 & 2 & 6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 0 & 10 \end{array}\right)$$

<u>row 3</u>: $0 = 10$. $\longrightarrow$ no solution inconsistent.

Ex. $\left(\begin{array}{ccc|c} 1 & 5 & 2 & 6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

<u>row 3</u>: $0 = 0$. $\rightarrow$ X_3 is free.

<u>row 2</u>: $4X_2 = 2 + 7X_3$.

$$\longrightarrow X_2 = \frac{1}{2} + \frac{7}{4}X_3.$$

<u>row 1</u>: $X_1 = -5X_2 - 2X_3 + 6$.

$$= -5\left(\frac{1}{2} + \frac{7}{4}X_3\right) - 2X_3 + 6$$

$$= \frac{7}{2} - \frac{43}{4}X_3.$$

$$\begin{cases} X_1 = \frac{7}{2} - \frac{43}{4}X_3 \\ X_2 = \frac{1}{2} + \frac{7}{4}X_3 \\ X_3 = \text{free} \end{cases} \longrightarrow \text{infinitely many solutions}$$

Ex.
$$\left(\begin{array}{ccc|c} 1 & 5 & 2 & -6 \\ -1 & -1 & -9 & 8 \\ 1 & 1 & 14 & -8 \end{array}\right)$$

Add row 1 to row 2.

Add $(-1)\times$row 1 to row 3.

$$\left(\begin{array}{ccc|c} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & -4 & 12 & -2 \end{array}\right)$$

Add row 2 to row 3.

$$\left(\begin{array}{ccc|c} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{array}\right)$$

row 3. $5X_3 = 0 \longrightarrow X_3 = 0$.

row 2 $4X_2 = 2 + 7X_3 = 2 \rightarrow X_2 = \frac{1}{2}$

row 1 $X_1 = -5X_2 - 2X_3 - 6 = \cancel{-10 - 0 - 6 = -16}$.

$$= -\frac{5}{2} - 0 - 6 = \frac{-17}{2}$$

Solution $= \left(\frac{-17}{2}, \frac{1}{2}, 0\right)$

Section 1.2. Row reduction and echelon form.

Terminology:

Consider a matrix.

"A non-zero row" = a row with at least one non-zero entry.

"A non-zero column".

"a row of all zeros".

"a column of all zeros"

"The leading entry of a non-zero row"

= "the left most non-zero entry.

Ex.

The leading entry of row 2

$$\begin{pmatrix} 0 & 7 & 2 & 3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a non-zero row

a row of all zeros.

a column of all zeros

non-zero column.