AMS 10/10A, Homework 10

Problem 1. Let H be a subspace spanned by $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $u_1 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$. Write $y = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$ as the sum of a vector in H and a vector orthogonal to H.

Problem 2. Find the closest point to y in the subspace spanned by v_1 and v_2, where

$$y = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \ v_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \text{ and } v_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

Problem 3-7. Let $A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ 2 & -4 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

- Is equation $Ax = b$ consistent?
- Verify that the two columns of matrix A form an orthogonal basis for $Col(A)$.
- Let $\hat{b}$ be the orthogonal projection of b onto $Col(A)$. Find $\hat{b}$.
- Let $\hat{x}$ the least square solution of $Ax = b$. Use the formula $\hat{x} = (A^T A)^{-1} A^T b$ to compute $\hat{x}$.
- Verify that $\hat{x}$ is the solution of $Ax = \hat{b}$.

Problem 8-9. Let A be an $m \times n$ matrix. Use the steps below to show that a vector x in $\mathbb{R}^n$ satisfies $Ax = 0$ if and only if $A^T Ax = 0$.

- Show that if $Ax = 0$, then $A^T Ax = 0$.
- Suppose $A^T Ax = 0$. Show that $x^T A^T Ax = 0$, and use this to prove $Ax = 0$.

Problem 10-11. Let A be an $m \times n$ matrix. Problem 8-9 implies that $Nul(A) = Nul(A^T A)$. Use this result to prove that

- $rank(A) = rank(A^T A)$.
- If $rank(A) = n$, then $A^T A$ is invertible.