01/11/2018

Recap

$$i = \sqrt{-1}$$

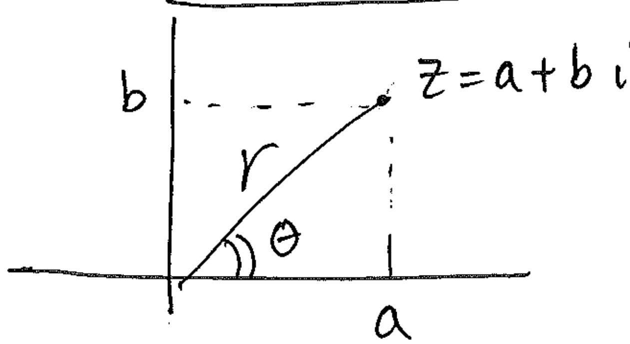
$$z = a + bi$$

Key step in calculating

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{z_2 \overline{z_2}} = \cdots$$

$$z \overline{z} = |z|^2$$

Geometry : complex plane



polar form: $r = |z|$

$$\theta = \arg(z)$$

$$z = r(\cos\theta + \sin\theta\, i)$$

z is completely specified by $|z|$ and $\arg(z)$

Multiplication

$$|z_1 z_2| = |z_1||z_2|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Conjugate

$$|\bar{z}| = |z|$$

$$\arg(\bar{z}) = -\arg(z)$$

Division

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

Exponential form

Taylor expansion of real exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Set $x = \theta i$

$$e^{\theta i} = 1 + \theta i - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} i + \frac{\theta^4}{4!}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)$$

$$+ \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) i$$

$$= \cos\theta + \sin\theta \, i$$

$$\boxed{e^{\theta i} = \cos\theta + \sin\theta \, i}$$

Polar form : $r(\cos\theta + \sin\theta \, i)$

Exponential form : $re^{\theta i}$

properties of e^z

*) $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$

*) $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

*) $\left(e^{z_1}\right)^{z_2} = e^{z_1 z_2}$

Multiplication in exponential form

$$z_1 = r_1 e^{\theta_1 i}$$

$$z_2 = r_2 e^{\theta_2 i}$$

$$z_1 z_2 = r_1 r_2 e^{(\theta_1+\theta_2) i}$$

Conjugate: $\overline{z_1} = r_1 e^{-\theta_1 i}$

Division: $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{\theta_1 i}}{r_2 e^{\theta_2 i}} = \dfrac{r_1}{r_2} e^{(\theta_1-\theta_2) i}$

Special cases of $e^{\theta i} = \cos\theta + \sin\theta\, i$

$$e^{2\pi i} = 1$$

$$e^{\pi i} = -1$$

$$e^{\frac{\pi}{2} i} = i$$

$$e^{\frac{3\pi}{2} i} = -i$$

$$e^{(\theta+2\pi)i} = e^{\theta i} \cdot e^{2\pi i} = e^{\theta i}$$

$$e^{(\theta+\pi)i} = e^{\theta i} \cdot e^{\pi i} = -e^{\theta i}$$

$$e^{(\theta+\frac{\pi}{2})i} = e^{\theta i} \cdot e^{\frac{\pi}{2} i} = i\, e^{\theta i}$$

$$e^{(\theta+\frac{3\pi}{2})i} = e^{\theta i} \cdot e^{\frac{3\pi}{2} i} = -i\, e^{\theta i}$$

θ	$\cos\theta$	$\sin\theta$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\vdots$		

Advantages of exp form

*). easy to do multi, div

*) easy to read out $|z|$, $\arg(z)$

$$z = \sqrt{3}\, e^{\frac{\pi}{3} i}$$

$$|z| = \sqrt{3} \qquad \arg(z) = \frac{\pi}{3}$$

$$z = r e^{\theta i}$$

$$\frac{1}{z} = \frac{1}{r} e^{-\theta i}$$

$$z^2 = r^2 e^{2\theta i}$$

Roots of polynomials

Theorem (Fundamental Theorem of Algebra): Consider

$$P(z) = C_n z^n + C_{n-1} z^{n-1} + \cdots + C_1 z + C_0$$

$n > 0$, $\{C_0, C_1, \ldots, C_n\}$ complex, $C_n \neq 0$.

Then there exists ζ_1 in $\mathbb{C}$ such that

$$P(\zeta_1) = 0$$

We factor $P(z)$ $\qquad \dfrac{P(z)}{z-\zeta_1}$

$$P(z) = (z-\zeta_1)\,P_2(z) + \alpha$$

$$0 = P(\zeta_1) = \underbrace{(\zeta_1-\zeta_1)\,P_2(\zeta_1)}_{=0} + \alpha$$

$$\Rightarrow \alpha = 0.$$

$$\begin{aligned} P(z) &= (z-\zeta_1)\,P_2(z) \\ &= (z-\zeta_1)(z-\zeta_2)\,P_3(z) \\ &= (z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_n)\,C_n. \end{aligned}$$

proposition 4.1-4.2

$P(z)$ can be writen as.

$$P(z) = (z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_n)\,C_n.$$

$\zeta_1, \zeta_2, \dots, \zeta_n$ are the roots of $P(z)$.

$P(z)$ has at least 1, at most n distinct roots polynomials of real coefficients

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

$\{a_0, a_1, \cdots, a_n\}$ real, $\quad a_n \neq 0$.

Ex: $\quad x^2+1 = 0$

no real root

$$x^2+1=0.$$
$$x^2-(-1)=0$$
$$x^2-i^2=0$$
$$(x-i)(x+i)=0$$

$x = i, -i \quad$ conjugate pair

proposition 4.3 If $P(\zeta)=0$, then $P(\bar{\zeta})=0$.

$$0 = P(\zeta) = a_n \zeta^n + \cdots + a_0$$
$$0 = \overline{P(\zeta)} = \overline{a_n \zeta^n + \cdots + a_0}$$
$$= \bar{a}_n (\bar{\zeta})^n + \cdots + \bar{a}_0$$
$$= a_n (\bar{\zeta})^n + \cdots + a_0 = P(\bar{\zeta})$$

$\Rightarrow P(\bar{\zeta}) = 0$

$$P(z) = (z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_n)$$

proposition 4.4 :

$$(z-\zeta)(z-\bar{\zeta})$$

$$= z^2 - z\underbrace{(\zeta+\bar{\zeta})}_{2\mathrm{Re}(\zeta)} + \underbrace{\zeta\bar{\zeta}}_{|\zeta|^2}$$

$$= z^2 - 2\mathrm{Re}(\zeta)\, z + |\zeta|^2$$

proposition 4.5 .

$P(z)$: polynomial of real coefficients.

$$P(z) = (z-\xi_1)(z-\xi_2)\cdots(z-\xi_m)\, Q_1(z)\cdots Q_K(z)$$

$\xi_1, \xi_2, \cdots, \xi_m$ are real roots

$\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2, \cdots, \zeta_K\, \bar{\zeta}_K$ are complex roots.

$$Q_j(z) = z^2 - 2\mathrm{Re}(\zeta_j)\, z + |\zeta_j|^2$$

Square roots of a complex number

let $u = re^{\theta i}$

A square root of u is a solution of $z^2 = u$

let $z = \rho e^{\omega i}$

$\Rightarrow (\rho e^{\omega i})^2 = re^{\theta i}$

$\Rightarrow \rho^2 e^{2\omega i} = re^{\theta i}$

$\Rightarrow \rho^2 = r \Rightarrow \rho = \sqrt{r}$

$\Rightarrow 2\omega = \theta + 2k\pi$

$\Rightarrow \omega = \frac{\theta}{2} + \frac{2k\pi}{2}$

$$= \begin{cases} \frac{\theta}{2}, & k=0 \\ \frac{\theta}{2}+\pi, & k=1 \\ \frac{\theta}{2}+2\pi \to \theta, & k=2 \end{cases}$$

$\boxed{\text{proposition 4.6}}$ $U = re^{\theta i}$ has two square roots.

$\zeta_1 = \sqrt{r}\, e^{\frac{\theta}{2} i}$, $\zeta_2 = \sqrt{r}\, e^{(\frac{\theta}{2}+\pi) i} = -\sqrt{r}\, e^{\frac{\theta}{2} i}$

n-th roots of a complex number

let $u = re^{\theta i}$

An n-th root of u is a solution of $z^n = u$

let $z = \rho e^{\omega i}$

$$(\rho e^{\omega i})^n = r e^{\theta i}$$

$$\rho^n e^{n\omega i} = r e^{\theta i}$$

$$\Rightarrow \rho^n = r \quad \Rightarrow \quad \rho = \sqrt[n]{r}$$

$$n\omega = \theta + 2K\pi$$

$$\omega = \frac{\theta}{n} + \frac{2K\pi}{n}.$$

$$= \begin{cases} \frac{\theta}{n} & K=0 \\ \frac{\theta+2\pi}{n} & K=1 \\ \vdots & \vdots \\ \frac{\theta+2(n-1)\pi}{n} & K=n-1 \\ \frac{\theta}{n}+2\pi \to \theta & K=n \end{cases}$$

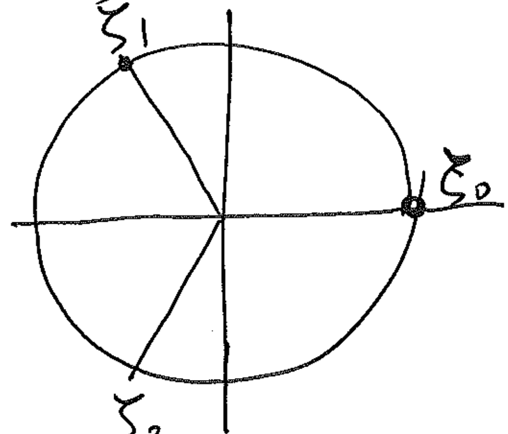
$\boxed{\text{proposition 4.8}}$: $z^n = re^{\theta i}$ has n ~~n-th roots~~ solutions.

$$z_K = \sqrt[n]{r}\, e^{\frac{\theta+2K\pi}{n} i}, \quad K = 0, 1, 2, \cdots, (n-1)$$

Ex. n-th roots of 1.

$$1 = 1 \cdot e^{0i} \qquad r=1, \quad \theta=0$$

$$\zeta_k = e^{\frac{2k\pi}{n}i}, \qquad k=0,1,2,\cdots,(n-1)$$

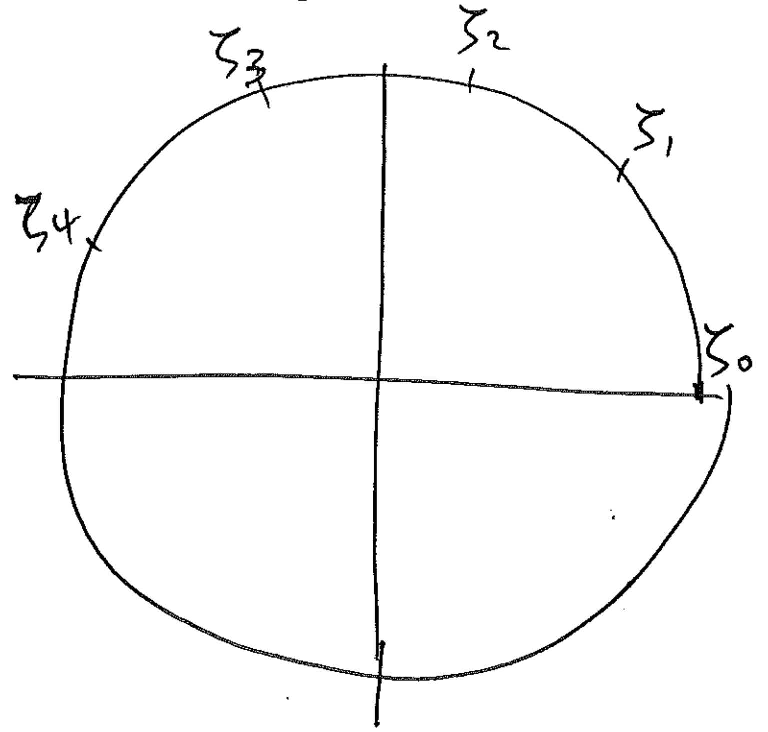


$\boxed{\text{Case of } n=3}$

$$\zeta_0 = e^{0i} = 1$$

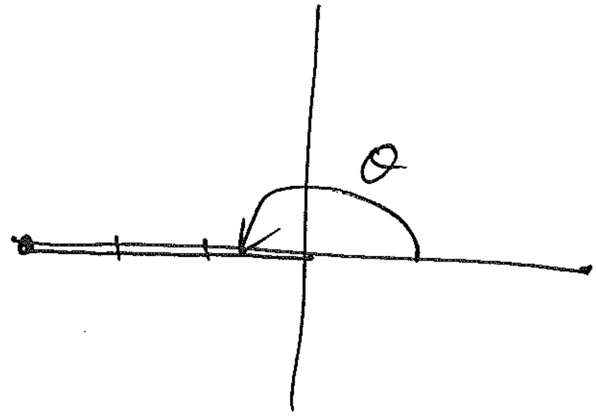
$$\zeta_1 = e^{\frac{2\pi}{3}i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\zeta_2 = e^{\frac{4\pi}{3}i} = e^{\frac{-2\pi}{3}i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$



Ex. Find all cubic roots of -3

$$-3 = 3e^{\pi i}$$

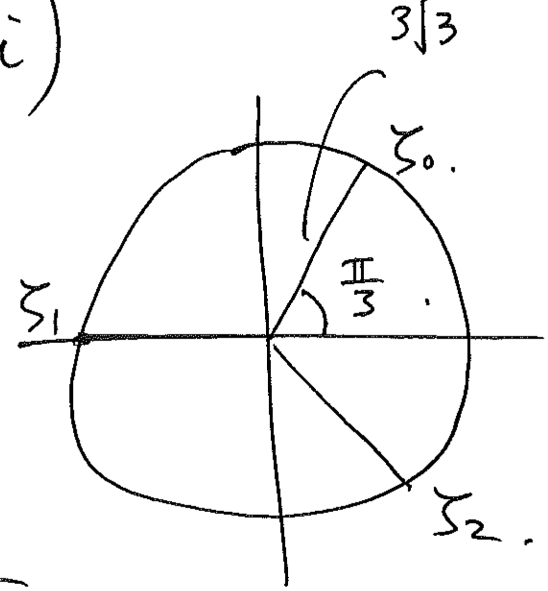


$$\zeta_0 = \sqrt[3]{r}\, e^{\frac{\theta}{3} i}$$

$$= \sqrt[3]{3}\, e^{\frac{\pi}{3} i} = \sqrt[3]{3}\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$

$$\zeta_1 = \sqrt[3]{r}\, e^{\frac{\theta + 2\pi}{3} i}$$

$$= \sqrt[3]{3}\, e^{\pi i} = -\sqrt[3]{3}.$$



$$\zeta_2 = \sqrt[3]{r}\, e^{\frac{\theta + 4\pi}{3} i}$$

$$= \sqrt[3]{3}\, e^{-\frac{\pi}{3} i} = \sqrt[3]{3}\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$$

Ex Find all square roots of $2i$

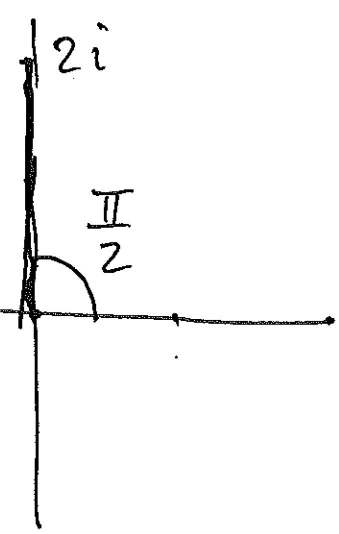
$$2i = 2e^{\frac{\pi}{2} i}$$

$r = 2 \quad \theta = \frac{\pi}{2}$

$$\zeta_0 = \sqrt{r}\, e^{\frac{\theta}{2} i}$$

$$= \sqrt{2}\, e^{\frac{\pi}{4} i} = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$$

$$= (1 + i)$$



$$\zeta_1 = \sqrt{r}\, e^{\frac{\theta+2\pi}{2} i}$$

$$= \sqrt{2}\, e^{\frac{5\pi}{4} i} = \sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right)$$

$$= -(1+i)$$

Ex. write $(\sin\alpha - \cos\alpha\, i)$ into <u>exp form</u>

Multiply by i $\quad 1 \cdot e^{\frac{\pi}{2} i}$

$$i(\sin\alpha - \cos\alpha\, i)$$

$$= \cos\alpha + \sin\alpha\, i = e^{\alpha i}$$

$$e^{\frac{\pi}{2} i}\, \underline{(\sin\alpha - \cos\alpha\, i)} = e^{\alpha i}$$

$$\Rightarrow (\sin\alpha - \cos\alpha\, i) = \frac{e^{\alpha i}}{e^{\frac{\pi}{2} i}} = e^{\left(\alpha - \frac{\pi}{2}\right) i}$$

$$\boxed{\sin\alpha - \cos\alpha\, i = e^{\left(\alpha - \frac{\pi}{2}\right) i}}$$