AMS 10

The special imaginary number

$$i = \sqrt{-1}$$

$$\boxed{i^2 = -1}$$

Other imaginary number

$$b\, i$$

real number

Background

$x^2 = -1$. has no solution. in real number.

Square root of a real number. $\underline{a}$

If $a > 0$ $\sqrt{a} = \sqrt{|a|}$

If $a < 0$. $\sqrt{a} = \sqrt{-|a|} = \sqrt{|a|}\sqrt{-1} = \sqrt{|a|}\, i$

Ex $\sqrt{-5} = \sqrt{5}\, i$

Roots of a quadratic eq.

$$ax^2 + bx + c = 0$$

<u>Two roots</u>

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

<u>Ex</u>. $x^2 + 2x + 5 = 0$

$$z_{1,2} = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

Complex numbers.

$$z = a + b\, \overset{\sqrt{-1}}{i}$$

real number

Equality

$$a_1 + b_1 i = a_2 + b_2 i$$

if and only if $a_1 = a_2$
$b_1 = b_2$

Notation: ~~Re(z)~~

$$z = a + bi$$

$$\mathrm{Re}(z) = a$$

$$\mathrm{Im}(z) = b.$$

$\mathbb{R} = \{$ ~~real~~ all real numbers $\}$.

$\mathbb{C} = \{$ all complex numbers $\}$.

Complex conjugate:

$$\overline{a+bi} = a - bi$$

Ex: $\overline{-2+3i} = -2-3i$

$\overline{a} = a$ if a is real

$\overline{i} = -i$

$\overline{bi} = -bi$

$\overline{\overline{z}} = z$

Arithmetic

Addition / Subtraction.

$$\underbrace{(a+bi)} + \underbrace{(c+di)} = (a+c) + (b+d)i$$

Commutativity:

$$u+v = v+u$$

Associativity $u+(v+w) = (u+v)+w$

So we can write $u+v+w$

<u>Ex</u> $z = a+bi$

$$z+\overline{z} = 2\,\mathrm{Re}(z)$$
$$z-\overline{z} = 2\,\mathrm{Im}(z)\,i$$

Multiplication

$$(a+bi)(c+di)$$
$$= ac + adi + bci + bd\overset{-1}{i^2}$$
$$= (ac-bd) + (ad+bc)i$$

Ex. $(-3+5i)(3+2i)$
$$= -19+9i$$

Commutativity. $u \cdot v = v \cdot u$

Distributivity: $u \cdot (v+w) = u \cdot v + u \cdot w$

Associativity: $u \cdot (v \cdot w) = (u \cdot v) \cdot w$.

We can write $u \cdot v \cdot w$

Ex: $4/2 \neq 2/4$

$$(8/4)/2 = 1$$

$$8/(4/2) = 4$$

We cannot write $8/4/2$

Division:

$$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1}.$$

$$z_2^{-1} = \frac{1}{z_2}$$

$$z_2 = a+bi$$

$$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)}$$

$$= \frac{a-bi}{a^2+b^2}$$

$$\frac{c+di}{a+bi} = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)}$$

$$= \frac{(ca+db)+(ad-bc)i}{a^2+b^2}$$

Ex. $$\frac{3+2i}{4-3i} = \frac{(3+2i)(4+3i)}{(4-3i)(4+3i)}$$

$$= \frac{(3\times4-2\times3)+(2\times4+3\times3)i}{4^2+3^2}$$

$$= \frac{6+17i}{25}$$

Arithmetic and Conjugate.

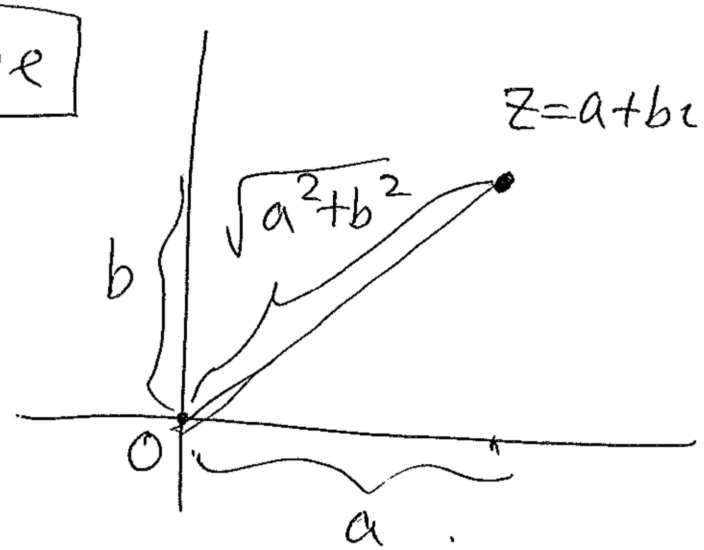
$$\overline{z+w} = \overline{z} + \overline{w}$$

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$$

The geometry of complex numbers.

$\boxed{\text{Complex plane}}$



$\boxed{\text{Absolute value}}$

$$|z| = |a+bi| = \sqrt{a^2+b^2}$$

Properties: $|\overline{z}| = |a-bi| = \sqrt{a^2+b^2} = |z|$.

$\boxed{1}$ $\quad |\overline{z}| = |z|$

$$z \cdot \overline{z} = (a+bi)(a-bi)$$
$$= a^2+b^2 = \left(\sqrt{a^2+b^2}\right)^2 = |z|^2$$

$\boxed{2}$: $z \cdot \overline{z} = |z|^2$

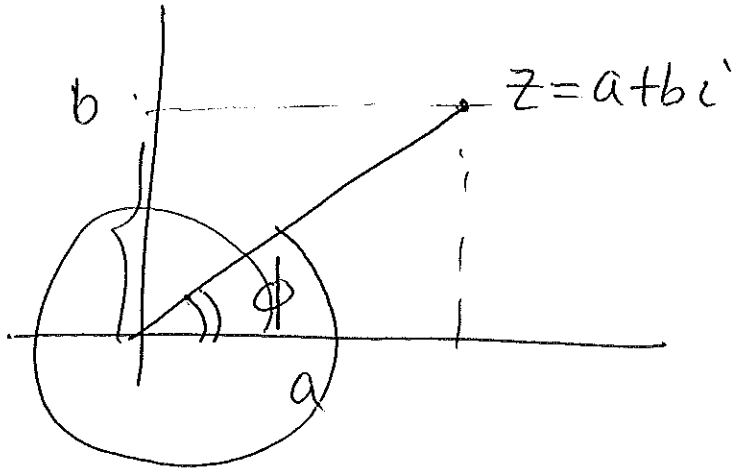
$$\frac{1}{z} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{\overline{z}}{|z|^2}$$

$\boxed{3}$ $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

$\boxed{4}$ $|z \cdot w| = |z| \cdot |w|$.

<u>proof</u> $|z \cdot w|^2 = (z \cdot w)\overline{(z \cdot w)}$
$$= z \cdot w \cdot \overline{z} \cdot \overline{w}$$
$$= \underline{z \cdot \overline{z}} \cdot \underline{w \cdot \overline{w}}$$
$$= |z|^2 \cdot |w|^2$$

The argument (phase) of a complex number



$$\underbrace{\arg(z)}_{\text{notation}} = \phi.$$

*). $\arg(z)$ is measured in radian.

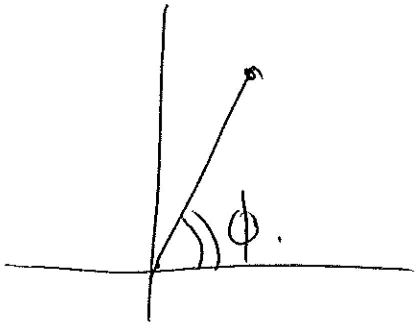
~~radian~~ $180° = \pi$ (in radian)

*). $\arg(z)$ is only determined up to an integer multiple of 2π.

*) The one in $[0, 2\pi)$. is called principle value.

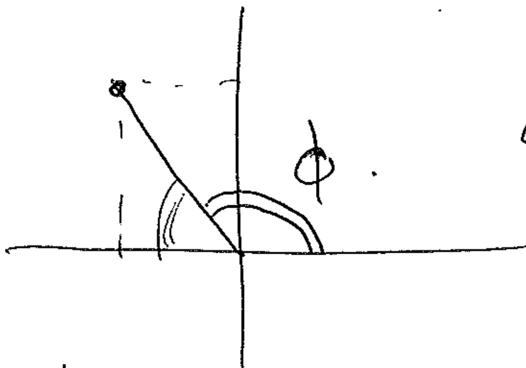
How to calculate the principle argument

*). First quadrant $a>0$, $b>0$, $z=a+bi$



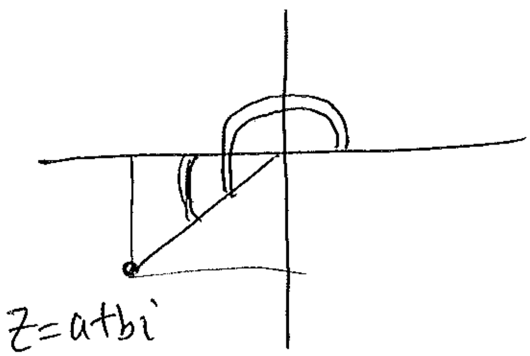
$\phi = \arg(z) = \operatorname{atan}\left(\frac{|b|}{|a|}\right)$.

*). Second quadrant. $a<0$, $b>0$. $z=a+bi$



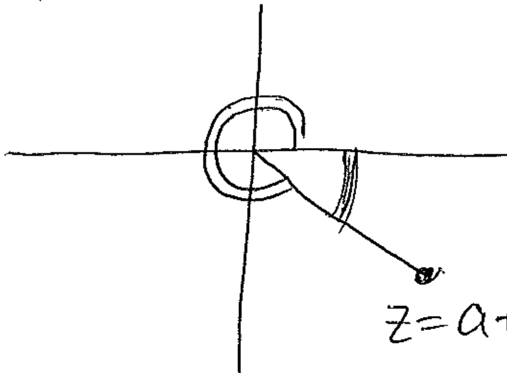
$\arg(z) = \pi - \operatorname{atan}\left(\frac{|b|}{|a|}\right)$

*) Third quadrant. $a<0$, $b<0$, $z=a+bi$



$\arg(z) = \pi + \operatorname{atan}\left(\frac{|b|}{|a|}\right)$

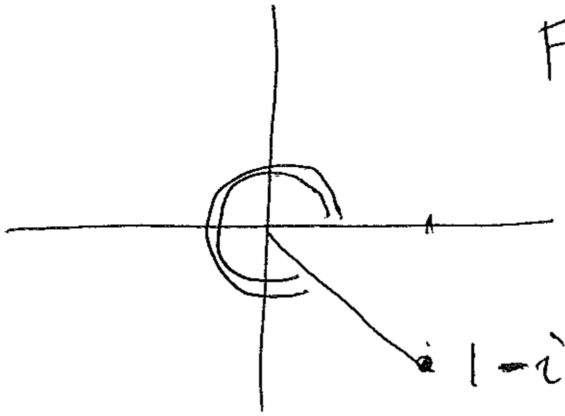
*) Fourth quadrant, $a>0$, $b<0$, $z=a+bi$



$\arg(z) = 2\pi - \operatorname{atan}\left(\frac{|b|}{|a|}\right)$.

Ex: $\arg(1-i)$

Fourth quadrant

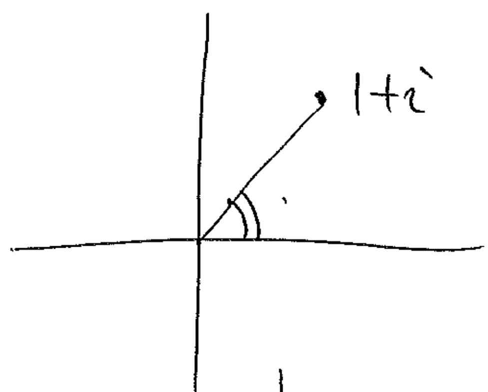


$a=1,\ b=-1.$

$$\arg(1-i) = 2\pi - \operatorname{atan}\left(\frac{1}{1}\right)$$

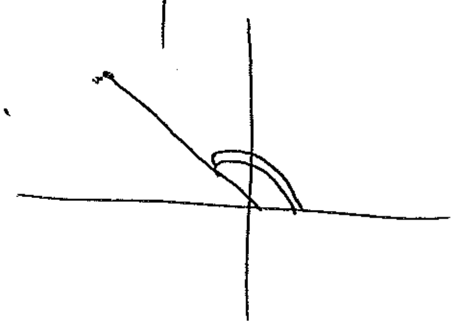
$$= 2\pi - \frac{\pi}{4} = \frac{7}{4}\pi.$$

$$\arg(1+i) = \frac{\pi}{4}$$

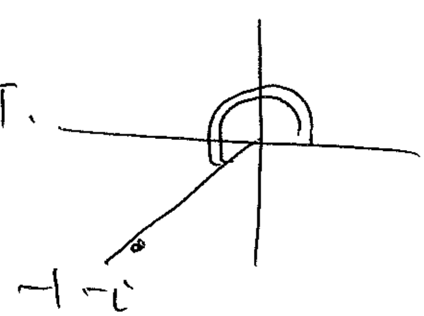


First quadrant

$$\arg(-1+i) = \frac{3}{4}\pi.$$

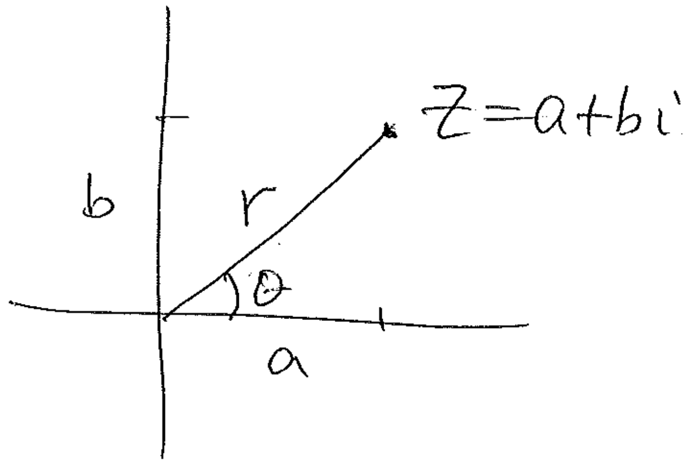


$$\arg(-1-i) = \frac{5}{4}\pi.$$



$z = a + bi$.

Cartesian form of a complex number.



$r = |z|$.

$\theta = \arg(z)$.

$a = r \cdot \cos\theta$

$b = r \cdot \sin\theta$

$$z = a + bi = r\cos\theta + r\sin\theta\, i$$

$$\boxed{z = r(\cos\theta + \sin\theta\, i)}$$

polar form of a complex number.

Multiplication

$z_1 = r_1(\cos\theta_1 + \sin\theta_1\, i)$

$z_2 = r_2(\cos\theta_2 + \sin\theta_2\, i)$

$$z_1 \cdot z_2 = r_1 \cdot r_2 \,\underline{(\cos\theta_1 + \sin\theta_1\, i)(\cos\theta_2 + \sin\theta_2\, i)}$$

$$= r_1 r_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + (\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) i \right]$$

$$= r_1 r_2 \left[\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2)\, i \right]$$

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|.$$

$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$

Conjugate: $z = r(\cos\theta + \sin\theta\, i)$

$$\bar{z} = r(\cos\theta - \sin\theta\, i)$$

$$= r(\cos(-\theta) + \sin(-\theta)\, i)$$

$$|\bar{z}| = |z|$$

$$\arg(\bar{z}) = -\arg(z)$$

Division. $z_1 = r_1(\cos\theta_1 + \sin\theta_1 i)$

$$z_2 = r_2(\cos\theta_2 + \sin\theta_2\, i)$$

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \cancel{\frac{r_1(\cos\theta_1 + \sin\theta_1 i) r_2 (\cos\theta_2}{r_2^2 \ |z_2|^2}}$$

$$= \frac{r_1 r_2 (\cos\theta_1 + \sin\theta_1 i)(\cos(-\theta_2) + \sin(-\theta_2) i)}{r_2^2}$$ (14)

$$= \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + \sin(\theta_1 - \theta_2) i\right)$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$