AMS 10/10A, Homework 1

Problem 1. Let $z_1 = 2 - 3i$ and $z_2 = -1 + 2i$. Compute

1. $\dfrac{z_1}{z_2}$;
2. $(z_1 + z_2)^3$;
3. $(z_1 - \bar{z}_2)^{-2}$.

Problem 2. Find a complex number z so that

$$(1 + 5i)z \quad = \quad -1 - 3i$$

Problem 3. Let $z_1 = \sqrt{3} + i$ and $z_2 = -1 - i$.

1. Express z_1 and z_2 using exponential notation, i.e., $z = re^{\theta i}$;
2. Compute $|z_1 \cdot z_2|$ and $\arg(z_1 \cdot z_2)$;
3. Compute $|z_1/z_2|$ and $\arg(z_1/z_2)$.

Problem 4. Express the following complex numbers using exponential notation, i.e., $z = re^{\theta i}$.

1. $\cos\alpha - i\sin\alpha$;
2. $\sin\alpha + i\cos\alpha$;
3. $\sin\alpha - i\cos\alpha$.

Hint: first use trigonometric identities to write each one in the form of $\cos\theta + i\sin\theta$.

Problem 5. Consider the following polynomial equation

$$x^4 \quad = \quad 2x$$

1. Find all real valued solutions of this equation;
2. Find all (real and complex) solutions of this equations.

Problem 6. Find all solutions of $z^3 = -1 + i$. Write out solutions using exponential notation.

Problem 7. Find all solutions of $z^4 = i$. Write out solutions using exponential notation.

Problem 8. Find all solutions of $z^4 = -1 + i\sqrt{3}$. Write out solutions using exponential notation.

Problem 9. Let $z = \sqrt{3} - i$.

1. Express z in the exponential notation;
2. Find the real and imaginary parts of z^{13}.
3. Find the real and imaginary parts of z^{22}.

Problem 10. In the complex plane, consider the set of all complex numbers satisfying

$$|z - (1 + i)| \;=\; 3$$

Identify the geometric meaning of the set.

Problem 11. Use exponential notation to show that multiplication by $\sqrt{2}+i\sqrt{2}$ corresponds to counterclockwise rotation by an angle of $\pi/4$ and stretching by a factor of 2.

Problem 12. Consider a polynomial of odd degree with real coefficients. Show that the polynomial always has at least one real root.
Hint: use Proposition 4.5 in the lecture notes; look at what happens to the degree of the right hand side if there is no real root.