AMS 10: Review for the Final Exam (Part 2)

Chapter 2 (continued)

The inverse of a matrix:

Definition of inverse:

Suppose matrix A is $n \times n$.

If there exists C such that C A = I and A C = I, then we say A is invertible.

Theorem 5 (of Chapter 2)

$$A\vec{x} = \vec{b} \xrightarrow{\text{A is invertible}} \vec{x} = A^{-1}\vec{b}$$

That is, invertibility of A implies existence and uniqueness of solution of $A\vec{x} = \vec{b}$.

If A and B are both invertible, then AB is invertible and

$(AB)^{-1}$ = ? (Theorem 6 of Chapter 2)

Theorem 7: (of Chapter 2)

Part 1: Matrix A is invertible if and only if A is row equivalent to I.

Part 2: Suppose $E_p \ldots E_2 E_1 A = I$.

Then we have $A^{-1} = E_p \ldots E_2 E_1 I$

The algorithm for finding A^{-1} (and determining if A is invertible)

(Theorem 7 of Chapter 2)

Theorem 8 of Chapter 2 (The invertible matrix theorem)

In particular, to show an $n \times n$ matrix A is invertible, we only need to find a matrix C such that C A = I

OR find a matrix C such that A C = I

Subspaces

Definition: 3 conditions

$H = span\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a subspace.

Col A

Nul A

A basis for a subspace

Definition: 2 conditions

How to find a basis for Col A? (Theorem 13, chapter 2)

How to find a basis for the subspace spanned by a set of vectors?

How to find a basis for Nul A?

Dimension of a subspace

How to find dim (Col A)?

How to find dim (Nul A)?

Rank of a matrix

How to find rank A?

<u>Theorem 13:</u>

The pivot columns of A form a basis for Col A.

<u>The rank theorem (Theorem 14):</u>

Suppose A is $m \times n$. We have

- rank A^T = rank A
- rank A + dim (Nul A) = n

<u>The basis theorem (Theorem 15):</u>

If the number of vectors in the given set matches the dimension of the subspace, we only need to check <u>one of the two conditions for a basis</u>.

Chapter 3

Determinant of a matrix (det A)

Co-factor expansion

Co-factor expansion along row i

Co-factor expansion along column j

Determinant of a 2 × 2 matrix

Determinant of a triangular matrix

Procedure of using ERO's to calculate det A

- Row reduce A to an echelon form
- Record and incorporate the effect of every ERO used

Properties of determinants:

A: $n \times n$

- $\det A \neq 0$ if and only if A is invertible (Theorem 4)
  if and only if columns of A are independent (Theorem 8, chapter 2)
  if and only if rows of A are independent (Theorem 8, chapter 2)
- $\det A^T = \det A$ (Theorem 5)
- $\det (AB) = (\det A)(\det B)$ (Theorem 6)
- $\det A^{-1} = \dfrac{1}{\det A}$
- $\det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn} \end{bmatrix} = a_{11} \cdots a_{nn}$
- $\det (\alpha A) = \alpha^n \det A$

Caution: in general, $\det(A+B) \neq (\det A) + (\det B)$

Chapter 5

Eigenvalues and eigenvectors of A

Definition:

λ is an eigenvalue of A if and only if $\det (A - \lambda I) = 0$.

Characteristic polynomial: $\det (A - \lambda I)$

Characteristic equation: $\det (A - \lambda I) = 0$

Algebraic multiplicity of an eigenvalue

Eigenspace of eigenvalue λ: $\text{Nul}\,(A - \lambda I)$

A basis for the eigenspace

Dimension of the eigenspace

Geometric multiplicity of an eigenvalue

Eigenvalues of a triangular matrix

Row reduction is not a tool for calculating eigenvalues of matrix A.

Theorem 2:

Eigenvectors for distinct eigenvalues are linearly independent.

Matrices A and B are similar if there exists an invertible matrix P such that

$B = P^{-1}AP$

Matrix A is diagonalizable if there exists an invertible matrix P such that

$P^{-1}AP = D$ where D is a diagonal matrix

Theorem 4:

A and B are similar

==> They have the same characteristic polynomial

==> the same set of eigenvalues and algebraic multiplicities

They also have the same geometric multiplicities

$$Nul\left(A-\lambda I\right) \underset{\text{One-to-one correspondence}}{\longleftrightarrow} Nul\left(B-\lambda I\right)$$

Theorem 5 of chapter 5: (The diagonalization theorem)

An $n \times n$ matrix is diagonalizable if and only if A has n linearly independent eigenvectors.

Theorem 6 of chapter 5:

If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Chapter 6

Inner product of two vectors

norm of a vector

distance between two vectors

orthogonal vectors

a vector orthogonal to a subspace

orthogonal complement of a subspace

orthogonal set

orthogonal basis

Theorem 2 (Pythagorean theorem):

$\vec{u} \perp \vec{v}$ if and only if $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

Properties of $W^{\perp} = \{\text{all } \vec{z} \text{ satisfying } \vec{z} \perp W\}$:

- $W^{\perp}$ is a subspace
- Let $W = span\{\vec{v}_1, \ldots, \vec{v}_p\}$.

  $\vec{x} \perp W$ if and only if $\vec{x} \perp \vec{v}_1$, $\vec{x} \perp \vec{v}_2$, …, $\vec{x} \perp \vec{v}_p$.

Theorem 3 of chapter 6:

A is an $m \times n$ matrix. We have

$$(\text{Row } A)^{\perp} = \text{Nul } A$$

$$(\text{Col } A)^{\perp} = \text{Nul } A^T$$

dim W and dim $W^{\perp}$

Let W be a subspace in $\mathbb{R}^n$. We have

$$\dim W + \dim W^{\perp} = n$$

Theorem 4 (of Chapter 6)

An orthogonal set of non-zero vectors is linearly independent.

Orthogonal projection of a vector onto a subspace

Theorem 8 (the orthogonal decomposition theorem)

Let W be a subspace in $\mathbb{R}^n$. Let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be an orthogonal basis for W.

Each $\vec{y}$ in $\mathbb{R}^n$ can be uniquely decomposed as

$\vec{y} = \vec{\hat{y}} + \vec{z}$ where $\vec{\hat{y}}$ is in W and $\vec{z}$ is in $W^{\perp}$.

$$\text{proj}_W \vec{y} \equiv \vec{\hat{y}} = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \cdots + \left(\frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\right)\vec{u}_p$$

Theorem 9 (the best approximation theorem)

Let W be a subspace in $\mathbb{R}^n$. Let $\vec{y}$ be a vector in $\mathbb{R}^n$.

We have

$\left\|\vec{y} - \text{proj}_W \vec{y}\right\| < \left\|\vec{y} - \vec{v}\right\|$ for all $\vec{v} \neq \text{proj}_W \vec{y}$ in W.

(Out of all vectors in W, $\text{proj}_W \vec{y}$ provides the unique best approximation to $\vec{y}$)

Question 15:

Suppose A and B are both $n \times n$ and are both invertible.

Is $(A^{-1}B^{-1})$ invertible? If so, what is $(A^{-1}B^{-1})^{-1}$?

Is $(B A^{-1})$ invertible? If so, what is $(B A^{-1})^{-1}$?

Question 16:

Suppose $A = [\vec{a}_1 \cdots \vec{a}_n]$ is an $n \times n$ matrix and is invertible.

Does matrix A have n pivot positions?

Does $A\vec{x} = \vec{0}$ have a non-trivial solution?

Is $\{\vec{a}_1, \ldots, \vec{a}_n\}$ linearly independent?

Is $A\vec{x} = \vec{b}$ consistent for every $\vec{b}$ in $\mathbb{R}^n$?

$span\{\vec{a}_1, \ldots, \vec{a}_n\} = \mathbb{R}^n$?

Hint: (Theorem 8 of Chapter 2)

Question 17:

Suppose A and B are both $n \times n$ and AB = I.

Is A invertible? If so, what is A^{-1}?

Is B invertible? If so, what is B^{-1}?

Hint: (Theorem 8 of Chapter 2)

Question 17B:

Suppose A and B are both $n \times n$ and AB = I.

What is BA ?

Hint: (Theorem 8 of Chapter 2)

Question 18:

Suppose A and B are both $n \times n$ and AB is invertible.

Is A invertible?

Is B invertible?

Is BA invertible?

Hint: (Theorem 8 of Chapter 2)

Question 19:

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & -3 & 6 \end{bmatrix}$

How do we determine whether or not matrix A is invertible

(without carrying out the full algorithm of finding the inverse)?

Hint: Row reduction to an echelon form and check the number of pivot positions.

Question 19B:

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & -3 & 6 \end{bmatrix}$

How do we find the inverse of matrix A?

Question 20:

How to check if a given vector $\vec{b}$ is in Col A?

Hint: check if $A\vec{x} = \vec{b}$ is consistent for the given $\vec{b}$.

Question 20B:

A: $m \times n$ matrix.

How to check if Col A = $\mathbb{R}^m$?

Hint: check if $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^m$.

Question 21:

How to check if $\vec{u}$ is in Nul A?

Hint: check whether or not $A\vec{u} = \vec{0}$.

Question 22:

How to find a basis for Nul A?

Hint: find the solution set of $A\vec{x} = \vec{0}$ in parametric form.

Question 23:

How to find a basis for Col A?

Hint: row reduction to echelon form to identify pivot columns. …

Question 23B:

How to find a basis for $\text{span}\left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right\}$?

Hint: find a basis for Col A where $A = \left[\vec{v}_1 \; \vec{v}_2 \; \cdots \; \vec{v}_n\right]$.

Question 24:

How to find dim (Col A)?

Hint: row reduction to echelon form to identify pivot columns …

Question 25:

How to find dim (Nul A)?

Hint: find the number of free variables in $A\vec{x} = \vec{0}$.

Question 26:

Suppose A is $m \times n$.

Is it true that rank A + dim (Nul A^T) = m?

Is it true that rank A^T + dim (Nul A) = n?

Hint: Use the rank theorem.

Question 26B:

Suppose A is $n \times n$ (square matrix).

Show that dim (Nul A) = dim (Nul A^T)

Hint: Use the rank theorem.

Question 26C:

Suppose A is 7×5.

Calculate dim (Nul A) – dim (Nul A^T)

Hint: Use the rank theorem.

Question 27:

Suppose an $n \times n$ matrix $A = \left[\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n \right]$ is invertible.

Is $\left\{ \vec{a}_1, \ldots, \vec{a}_n \right\}$ a basis for $\mathbb{R}^n$?

Col A = $\mathbb{R}^n$?

dim Col A = n ?

rank A = n ?

dim Nul A = 0 ?

det A = 0 ?

Is 0 an eigenvalue of A ?

Hint: Use the invertible matrix theorem.

Question 28:

$$\text{Find} \quad \det \begin{bmatrix} 2 & 2 & 5 & 4 & 1 & 7 \\ 1 & 1 & 3 & 6 & 8 & 1 \\ 5 & 5 & 2 & 7 & 7 & 6 \\ 7 & 7 & 6 & 1 & 5 & 4 \\ 3 & 3 & 7 & 5 & 2 & 3 \\ 1 & 1 & 4 & 3 & 4 & 5 \end{bmatrix}$$

Hint: Notice that column 1 and column 2 are the same.

Question 29:

Suppose matrix A is 5×5 and det A = 3. Find

det A^{-1}

det A^T

det A^2

det $(A^T A)$

det $(A A^T)$

det $(A^3 A^T)$

$\det\left(\frac{1}{3}A\right)$

det (A+A)

det $(A^T + A^T)$

Hint: Use properties of determinants.

Question 30:

How to find eigenvalues of A?

Hint: Solve the characteristic equation det $(A-\lambda I) = 0$.

Question 31:

Find the eigenvalues of $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn} \end{bmatrix}$.

Question 32:

How to find a basis for the eigenspace of eigenvalue λ?

Hint: Find a basis for Nul $(A-\lambda I)$.

Question 32B:

How to find the geometric multiplicity of eigenvalue λ?

Hint: find the number of free variable in $(A-\lambda I)\vec{x} = \vec{0}$.

Question 33:

How to find the algebraic multiplicity of eigenvalue λ?

Hint: Factor the characteristic polynomial det $(A-\lambda I)$.

Question 34:

How to read out eigenvalues and eigenvectors from the given diagonalization

$P^{-1}AP = D$ where matrix D is diagonal?

Hint:

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \qquad \lambda_1, \lambda_2, \ldots, \lambda_n \text{ are eigenvalues}$$

$$P = \left[\vec{v}_1 \cdots \vec{v}_n \right], \qquad \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \text{ are corresponding eigenvectors}$$

Question 35: How to diagonalize A?

Hint:

Step 1: find eigenvalues

Step 2: find a basis for each eigenspace

Steps 3 and 4: construct D and P

Question 36:

How to find a basis for $(\text{Col}\, A)^{\perp}$?

Hint: We use $(\text{Col}\, A)^{\perp} = \text{Nul}\, A^T$. We find a basis for Nul A^T.

Question 37:

Let $W = span\left\{ \vec{v}_1, \ldots, \vec{v}_p \right\}$.

How to find a basis for $W^{\perp}$?

Hint:

Let $A=\left[\vec{v}_1 \cdots \vec{v}_p\right]$. We have $W^{\perp}=\left(\text{Col}\,A\right)^{\perp}=\text{Nul}\,A^T$ ...

<u>Question 38:</u>

Suppose $\left\{\vec{v}_1,\ldots,\vec{v}_n\right\}$ is an <u>orthogonal</u> set of *n* <u>non-zero</u> vectors in R^n.

Is $\left\{\vec{v}_1,\ldots,\vec{v}_n\right\}$ linearly independent?

Is $\left\{\vec{v}_1,\ldots,\vec{v}_n\right\}$ a basis for R^n?

Is $\left\{\vec{v}_1,\ldots,\vec{v}_n\right\}$ an orthogonal basis for R^n?

Is $\text{span}\left\{\vec{v}_1,\ldots,\vec{v}_n\right\}=\mathbb{R}^n$?

<u>Question 39:</u>

Let W be a subspace in R^n. Let $\left\{\vec{u}_1,\ldots,\vec{u}_p\right\}$ be an orthogonal basis for W.

Vector $\vec{x}$ in W is a linear combination of $\left\{\vec{u}_1,\ldots,\vec{u}_p\right\}$.

Write out the coefficients in the linear combination using inner products.

<u>Question 40:</u>

Let W be a subspace in R^n. Let $\left\{\vec{u}_1,\ldots,\vec{u}_p\right\}$ be an orthogonal basis for W.

Write out the projection of vector $\vec{y}$ in $\mathbb{R}^n$ onto subspace W

$$\underbrace{\text{proj}_W\,\vec{y}}_{\substack{\text{Projection of}\\ \vec{y}\text{ onto W}}} = ?$$