AMS 10 Midterm Exam **Winter 2018**

Your Name: ____

Your Student ID #: ________________________________

Exam Scores:

Problem 1 (10 points)	
Problem 2 (10 points)	
Problem 3 (10 points)	
Problem 4 (10 points)	
Problem 5 (10 points)	
Total (50 points)	

Values of the trigonometric functions:

θ	Cos(θ)	Sin(θ)
$\pi/6$	$\sqrt{3}/2$	$1/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	1

Problem 1 (10 points): Let $z = \frac{1+\sqrt{3}\,i}{2}$.

a. Find the <u>real part</u> and the <u>imaginary part</u> of z^{35}. (4 points)
(you may need the table on the cover page).

$$z = \frac{1+\sqrt{3}\,i}{2} = \cos\frac{\pi}{3} + \sin\frac{\pi}{3}\,i = e^{i\frac{\pi}{3}}$$

$$\Longrightarrow \quad z^{35} = e^{i\frac{\pi}{3}\times 35} = e^{i\left(10\pi - \frac{\pi}{3}\right)} = e^{-i\frac{\pi}{3}} = \cos\left(\frac{-\pi}{3}\right) + \sin\left(\frac{-\pi}{3}\right) i = \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$\Longrightarrow \quad \operatorname{Re}\left(z^{35}\right) = \frac{1}{2}, \qquad \operatorname{Im}\left(z^{35}\right) = \frac{-\sqrt{3}}{2}$$

b. Write out ALL 5-th roots of z in the exponential form. (6 points)

z has the exponential form $z = e^{i\theta}$, $\theta = \frac{\pi}{3}$

The 5-th roots of z are

$$\zeta_0 = e^{i\left(\frac{\theta}{5}\right)} = e^{i\frac{\pi}{15}}$$

$$\zeta_1 = e^{i\left(\frac{\theta+2\pi}{5}\right)} = e^{i\frac{7\pi}{15}}$$

$$\zeta_2 = e^{i\left(\frac{\theta+4\pi}{5}\right)} = e^{i\frac{13\pi}{15}}$$

$$\zeta_3 = e^{i\left(\frac{\theta+6\pi}{5}\right)} = e^{i\frac{19\pi}{15}}$$

$$\zeta_4 = e^{i\left(\frac{\theta+8\pi}{5}\right)} = e^{i\frac{25\pi}{15}}$$

Problem 2 (10 points): Let $A = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 0 & 0 & -2 & 3 \\ 2 & -4 & 4 & 2 \end{bmatrix}$.

a. Solve $A\vec{x} = \vec{0}$ and write the solution set in parametric form. (6 points)

$$A = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 0 & 0 & -2 & 3 \\ 2 & -4 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 1 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 0 & 1 & -3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns: Col 1 and Col 3

Basic variables: x_1 and x_3

Free variables: x_2 and x_4

Solution set in parametric form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_4 \\ x_2 \\ (3/2)x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 3/2 \\ 1 \end{bmatrix}$$

b. Is $A\vec{x} = \vec{b}$ consistent for every $\vec{b}$ in $\mathbb{R}^3$? <u>And why?</u> (4 points)

$A\vec{x} = \vec{b}$ is not consistent for every $\vec{b}$ in $\mathbb{R}^3$

because matrix A does not have a pivot position in every row.

Problem 3 (10 points): Let $A = \begin{bmatrix} -7 & 5 & 1 & -4 & 2 & 6 \\ 2 & 1 & 3 & 5 & -7 & -2 \\ 4 & 3 & -5 & 3 & 1 & -3 \\ 3 & -2 & 7 & -2 & 4 & 5 \end{bmatrix}$

a. Are the columns of matrix A linearly independent? And why? (5 points)

There are 6 columns. There are 4 entries in each column.

# of columns > # of entries in each column

==> The columns of A are not linearly independent. They are linearly dependent.

b. Does $A\vec{x} = \vec{0}$ have a non-trivial solution? And why? (5 points)

The columns of A are linearly dependent.

==> $A\vec{x} = \vec{0}$ has a non-trivial solution.

Problem 4 (10 points): Let $A = \begin{bmatrix} 2 & 9 & 5 & 1 \\ 5 & -3 & 1 & 4 \\ 7 & 1 & 3 & -2 \\ 3 & 7 & 4 & 1 \\ 4 & 2 & 7 & 3 \end{bmatrix}$

a. Do the columns of A span $\mathbb{R}^5$? <u>And why</u>? (5 points)

There are 4 columns.

Each column has at most one pivot position.

==> # of pivot positions ≤ 4.

There are 5 rows.

==> Matrix does not have a pivot position in every row.

==> The columns of A does not span $\mathbb{R}^5$.

b. Compute the (3, 2) entry of A A^T (3 points)

We use the row-vector rule to calculate the (3, 2) entry of A A^T.

$$\left(AA^T\right)_{3,2} = row_3(A) \cdot col_2\left(A^T\right) = \begin{bmatrix} 7 & 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 1 \\ 4 \end{bmatrix}$$

$$= 7 \times 5 + 1 \times (-3) + 3 \times 1 + (-2) \times 4 = 27$$

c. How many rows and how many columns does A^T(A A^T) have? (2 points)

A: 5 × 4

A A^T: 5 × 5

A^T: 4 × 5

A^T(A A^T): 4 × 5

A^T(A A^T) has 4 rows and 5 columns.

Problem 5 (10 points): Let $A=\begin{bmatrix} 1 & -2 & 1 \\ 2 & -2 & -6 \\ -4 & 5 & 9 \\ -1 & 1 & 3 \end{bmatrix}$.

a. Is $A\vec{x}=\begin{bmatrix} 0 \\ 8 \\ -9 \\ -4 \end{bmatrix}$ consistent? If so, find a solution. (3 points)

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -2 & -6 & 8 \\ -4 & 5 & 9 & -9 \\ -1 & 1 & 3 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \\ 0 & -1 & 4 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The rightmost column of the augmented matrix is not a pivot column.

It is consistent and has a solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 29 \\ 16 \\ 3 \end{bmatrix}$.

b. Is $A\vec{x}=\vec{b}$ consistent for every $\vec{b}$ in $\mathbb{R}^4$? <u>And why</u>? (3 points)

Matrix A has 3 pivot positions and 4 rows.

==> Matrix A does not have a pivot column in every row.

==> $A\vec{x}=\vec{b}$ is not consistent for every $\vec{b}$ in $\mathbb{R}^4$.

c. Does $A\vec{x}=\vec{0}$ have a non-trivial solution? <u>And why</u>? (2 points)

All 3 columns of Matrix A are pivot columns.

==> All variables are basic variables. There is no free variable.

==> $A\vec{x}=\vec{0}$ does not have a non-trivial solution.

d. Are the columns of matrix A linearly independent? <u>And why</u>? (2 points)

$A\vec{x}=\vec{0}$ has only the trivial solution.

==> The columns of matrix A are linearly independent.