AMS 10/10A, Homework 9 Solutions

Problem 1. Since A and B are similar, there is an invertible matrix P such that $A = P^{-1}BP$.

$$
\begin{aligned}
A^2 &= (PBP^{-1})(PBP^{-1}) = PB(P^{-1}P)BP^{-1} = PB^2P^{-1} \\
A^3 &= A^2A = (PB^2P^{-1})(PBP^{-1}) = PB^3P^{-1} \\
&\vdots \\
A^k &= A^{k-1}A = (PB^{k-1}P^{-1})(PBP^{-1}) = PB^kP^{-1}
\end{aligned}
$$

Problem 2.

$$
\begin{aligned}
\lambda_1 &= 3, \qquad v_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \\
\lambda_2 &= 4, \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
\lambda_3 &= 3, \qquad v_3 = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}
\end{aligned}
$$

Problem 3.

Matrix A is not diagonalizable, since eigenvalue 3 has algebraic multiplicity 2 but geometric multiplicity 1.

Eigenvalues and bases for the eigenspaces of matrix B are

$$
\begin{aligned}
\lambda_1 &= -5, \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\lambda_2 &= 1, \qquad \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 \\ 0 \\ 6 \\ 0 \end{bmatrix} \\
\lambda_2 &= 3, \qquad \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}
\end{aligned}
$$

Matrix B can be diagonalized as

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-1} B \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Problem 4. Matrix A is not diagonalizable. Matrix A needs 9 linearly independent eigenvectors for it to be diagonalizable. But there exist only $3+3+2=8$ linearly independent eigenvectors.

Problem 5. The matrix is diagonalizable, since all its eigenvalues are distinct.

Problem 6. Prove that if A is both diagonalizable and invertible, then A^{-1} is also diagonalizable and invertible.

Proof: If A is both diagonalizable and invertible, there exist invertible matrix P and diagonal matrix D, such that $A = PDP^{-1}$. Since $A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$ and D^{-1} is diagonal, A^{-1} is diagonalizable.

Problem 7. Prove that if A is diagonalizable, A^k is also diagonalizable for any positive integer k.

Proof: If A is diagonalizable, there exist invertible matrix P and diagonal matrix D, such that $A = PDP^{-1}$. Then

$$A^k \; = \; (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) \; = \; PD^kP^{-1}$$

and D^k is always diagonal. Hence A^k is diagonalizable.

Problem 8.

$$u^Tv \; = \; -1, \qquad v^Tu \; = \; -1, \qquad \left(\frac{u^Tu}{v^Tu}\right) u \; = \; \begin{bmatrix} -14 \\ -42 \\ 28 \end{bmatrix}, \qquad \|u-v\| \; = \; \sqrt{33}$$

Problem 9. The first pair of vectors are orthogonal. The second pair of vectors are not orthogonal. The third pair of vectors are orthogonal.

Problem 10. Prove the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 \; = \; 2\|u\|^2 + 2\|v\|^2$$

where u and v are vectors in $\mathbb{R}^n$.

Proof:

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= (u+v)^T(u+v) + (u-v)^T(u-v) \\
&= (u^Tu + v^Tv + u^Tv + v^Tu) + (u^Tu + v^Tv - u^Tv - v^Tu) \\
&= 2u^Tu + 2v^Tv \\
&= \|u\|^2 + \|v\|^2
\end{aligned}$$

Problem 11. Suppose a vector x is orthogonal to vectors y and z. Prove that x is orthogonal to any vector in $span\{y, z\}$.

Proof: Any vector, v, in $span\{y, z\}$ is a linear combination of y and z, i.e., there exist coefficients c_1 and c_2 such that $v = c_1y + c_2z$.

$$\begin{aligned}
x^Tv &= x^T(c_1y + c_2z) \\
&= c_1x^Ty + c_2x^Tz \\
&= 0
\end{aligned}$$

Problem 12.

$$H^\perp = Nul(A^T) = span\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

Problem 13. Since H is a subspace in $\mathbb{R}^3$ with dimension 1, we have

$$dim(H^\perp) = 3 - dim(H) = 3 - 1 = 2$$

Problem 14. $Col(A)$ is a subspace in $\mathbb{R}^7$. It follows that

$$\begin{aligned}
& dim\ [Col(A)]^\perp + dim\ Col(A) = 7 \\
\Rightarrow\quad & dim\ [Col(A)]^\perp = 7 - dim\ Col(A)
\end{aligned}$$

On the other hand, $dim\ Col(A) \leq min\{5, 7\} = 5$. Therefore, we have

$$dim\ [Col(A)]^\perp \geq 7 - 5 = 2$$

The smallest possible dimension of $Col(A)^\perp$ is 2.

Problem 15. Set 1 is orthogonal. Set 2 is not.

Problem 16.

- Since $u_1^T u_2 = u_1^T u_3 = u_2^T u_3 = 0$, $\{u_1, u_2, u_3\}$ is an orthogonal set of 3 non-zero vectors in $\mathbb{R}^3$. Therefore, it is an orthogonal basis for $\mathbb{R}^3$.

- The representation of x is

$$\begin{aligned} x &= \frac{x^T u_1}{u_1^T u_1} u_1 + \frac{x^T u_2}{u_2^T u_2} u_2 + \frac{x^T u_2}{u_2^T u_2} u_2 \\ &= \frac{4}{3} u_1 + \frac{2}{9} u_2 + \frac{5}{9} u_3 \end{aligned}$$

Problem 17.

$$\begin{aligned} v &= \left(\frac{v^T u_1}{u_1^T u_1} u_1 + \frac{v^T u_2}{u_2^T u_2} u_2 \right) + \left(\frac{v^T u_3}{u_3^T u_3} u_3 + \frac{v^T u_4}{u_4^T u_4} u_4 \right) \\ &= \left(u_1 - \frac{5}{7} u_2 \right) + \left(\frac{8}{7} u_3 - \frac{3}{7} u_4 \right) \\ &= \begin{bmatrix} 17/7 \\ 9/7 \\ 12/7 \\ 2/7 \end{bmatrix} + \begin{bmatrix} 11/7 \\ 5/7 \\ -19/7 \\ -2/7 \end{bmatrix} \end{aligned}$$