AMS 10/10A, Homework 8 Solutions

Problem 1. v is an eigenvector since

$$Av = \begin{bmatrix} 4 \\ -4 \\ 16 \end{bmatrix} = 4v$$

Problem 2. $\lambda = 1$ is an eigenvalue since $det(A - I) = 0$.

Problem 3:

- Eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 9$.
  For $\lambda_1 = 2$ a corresponding eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
  For $\lambda_1 = 9$ a corresponding eigenvector is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

- Eigenvalues of B are $\lambda_1 = 1$ and $\lambda_2 = 9$.
  For $\lambda_1 = 2$ a corresponding eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
  For $\lambda_1 = 9$ a corresponding eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

- Eigenvalues of C are $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = 3$.
  For $\lambda_1 = 1$ a corresponding eigenvector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.
  For $\lambda_1 = -1$ a corresponding eigenvector is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.
  For $\lambda_1 = 3$ a corresponding eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Problem 4: The characteristic polynomial is

$$det(A - \lambda I) = \lambda^2 - (a + c)\lambda + ac - b^2$$

For the characteristic equation $\lambda^2 - (a + c)\lambda + ac - b^2 = 0$, we have

$$(a + c)^2 - 4(ac - b^2) = a^2 + c^2 - 2ac + 4b^2 = (a - c)^2 + 4b^2 \geq 0$$

Therefore, the eigenvalues cannot be complex.

Problem 5: The characteristic polynomial is

$$\begin{aligned} det(A - \lambda I) &= det \begin{bmatrix} \sin(\theta) - \lambda & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) - \lambda \end{bmatrix} \\ &= \lambda^2 - 2\sin(\theta)\lambda + 1 \end{aligned}$$

The characteristic equation $\lambda^2 - 2\sin(\theta)\lambda + 1 = 0$ has two solutions:

- $\lambda_1 = \sin(\theta) + \cos(\theta)i$, a corresponding eigenvector $\begin{bmatrix} -i \\ 1 \end{bmatrix}$
- $\lambda_2 = \sin(\theta) - \cos(\theta)i$, a corresponding eigenvector $\begin{bmatrix} i \\ 1 \end{bmatrix}$

Problem 6: Since λ is an eigenvalue of A, there exists a nonzero vector v such that $Av = \lambda v$. Since

$$\begin{aligned} A^k v &= A^{k-1}(Av) = A^{k-1}(\lambda v) = \lambda(A^{k-1}v) \\ &= \lambda A^{k-2}(Av) = \lambda A^{k-2}(\lambda v) = \lambda^2 A^{k-2} v \\ &= \cdots = \lambda^k v \end{aligned}$$

λ^k is an eigenvalue of A^k.

Problem 7. If λ is an eigenvalue of an invertible matrix A, λ must be nonzero.

$$\begin{aligned} & Av = \lambda v \\ \Longrightarrow\ & v = A^{-1}(\lambda v) \\ \Longrightarrow\ & \lambda^{-1} v = A^{-1} v \end{aligned}$$

Therefore, λ^{-1} is an eigenvalue of A^{-1}.

Problem 8. The characteristic polynomial of A^T is

$$det(A^T - \lambda I) = det((A - \lambda I)^T) = det(A - \lambda I)$$

Therefore, A and A^T have the same eigenvalues.

Problem 9.

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = a_1 + a_2 + \cdots + a_n = \begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Therefore, s is an eigenvalue and $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is a corresponding eigenvector.

Problem 10. Since $c_1v_1 + c_2v_2$ is not the zero vector and satisfies

$$\begin{aligned} A(c_1v_1 + c_2v_2) &= c_1Av_1 + c_2Av_2 = c_1\lambda v_1 + c_2\lambda v_2 \\ &= \lambda(c_1v_1 + c_2v_2) \end{aligned}$$

it is an eigenvector of A corresponding to λ.

Problem 11. The eigenvalues and their multiplicities are

- $\lambda_1 = 3$, with algebraic multiplicity 1 and geometric multiplicity 1;
- $\lambda_2 = -2$, with algebraic multiplicity 1 and geometric multiplicity 1;
- $\lambda_3 = 1$, with algebraic multiplicity 2 and geometric multiplicity 1;

Problem 12.

$$A - 4I = \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & \alpha & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The echelon form of $A - 4I$ is

$$\begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & 0 & \alpha + 3 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

When $\alpha = -3$, there are two free variables in homogeneous equation $(A - 4I)x = 0$. Therefore, when $\alpha = -3$ the geometric multiplicity of $\lambda = 4$ is 2.

Problem 13.

$$\lambda_1 = 3, \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 8 + 2i, \qquad v_2 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

$$\lambda_3 = 8 - 2i, \qquad v_3 = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Problem 14. The characteristic polynomial is

$$\begin{aligned} det(A - \lambda I) &= det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \\ &= \lambda^2 - \lambda(a + d) + (ad - bc) \end{aligned}$$

Let λ_1 and λ_2 be the two solutions of the characteristic equation

$$\lambda^2 - \lambda(a + d) + (ad - bc) = 0$$

The characteristic equation can be written as

$$\begin{aligned} & (\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \\ \Longrightarrow \quad & \lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1 \cdot \lambda_2 = 0 \end{aligned}$$

Comparing the two expressions for the characteristic polynomial, we conlcude

$$\lambda_1 \cdot \lambda_2 = ad - bc = det(A)$$

Problem 15. Comparing the two expressions for the characteristic polynomial in Problem 14, we conlcude

$$\lambda_1 + \lambda_2 = a + d$$