AMS 10/10A, Homework 7 Solutions

Problem 1. Suppose A is $m \times n$. Prove the following equality

$$dim\ Col(A) + dim\ Nul(A^T) \;\; = \;\; m$$

Proof: Matrix A^T is $n \times m$. By the Rank Theorem,

$$rank\ A^T + dim\ Nul(A^T) \;\; = \;\; m$$

Also, $rank\ A^T = rank\ A = dim\ Col(A)$. Therefore, we conclude

$$dim\ Col(A) + dim\ Nul(A^T) = m$$

Problem 2. Suppose A is $m \times n$ and b is in R^m. Prove that if the equation $Ax = b$ is consistent, then $rank\ [A, b] = rank\ A$.
Proof: If the equation $Ax = b$ is consistent, then b is a linear combination of the columns of A. That is, b is a linear combination $\{a_1, a_2, \cdots, a_n\}$ where $a_1, a_2, \cdots, a_n$ are columns of A. Therefore, we have

$$Col\ [A, b] \;\; = \;\; span\{a_1, a_2, \cdots, a_n, b\} \;\; = \;\; span\{a_1, a_2, \cdots, a_n\} \;\; = \;\; Col\ A$$

It follows that

$$rank\ [A, b] \;\; = \;\; dim\ Col\ [A, b] \;\; = \;\; dim\ Col\ A \;\; = \;\; rank\ A$$

Problem 3:

- Yes. $Ax = 0$ has non-trivial solutions. $rank\, A = 5$ means that matrix A has 5 pivot columns. $Ax = 0$ has 8 variables. So $Ax = 0$ has 5 basic variables and 3 free variables.
- No. $A^T x = 0$ does not have a non-trivial solution. $rank\, A = 5$ means that $rank\, A^T = 5$ and matrix A^T has 5 pivot columns. $A^T x = 0$ has 5 variables. So $A^T x = 0$ has 5 basic variables and no free variable.

Problem 4: $det(A) = 0$, $det(B) = 107$, $det(C) = -48$, $det(D) = 48$.

Problem 5:

$$det\begin{bmatrix} a & 0 & d & c \\ b & 0 & -c & d \\ 0 & c & -b & a \\ 0 & d & a & b \end{bmatrix} = a\cdot det\begin{bmatrix} 0 & -c & d \\ c & -b & a \\ d & a & b \end{bmatrix} - b\cdot det\begin{bmatrix} 0 & d & c \\ c & -b & a \\ d & a & b \end{bmatrix}$$

$$= a\left(-c\cdot det\begin{bmatrix} -c & d \\ a & b \end{bmatrix} + d\cdot\begin{bmatrix} -c & d \\ -b & a \end{bmatrix}\right)$$

$$-b\left(-c\cdot det\begin{bmatrix} d & c \\ a & b \end{bmatrix} + d\cdot\begin{bmatrix} d & c \\ -b & a \end{bmatrix}\right)$$

$$= a\left(-c(-cb-ad)+d(-ca+bd)\right)$$

$$-b\left(-c(db-ac)+d(da+bc)\right)$$

$$= ac^2b + a^2cd - a^2cd + abd^2 + b^2cd - ac^2b - abd^2 - b^2cd$$

$$= 0$$

Problem 6:

$$det\begin{bmatrix} a & \sqrt{2} & 0 \\ \sqrt{2} & a & \sqrt{2} \\ 0 & \sqrt{2} & a \end{bmatrix} = a\cdot det\begin{bmatrix} a & \sqrt{2} \\ \sqrt{2} & a \end{bmatrix} - \sqrt{2}\cdot det\begin{bmatrix} \sqrt{2} & 0 \\ \sqrt{2} & a \end{bmatrix}$$

$$= a(a^2-2) - \sqrt{2}(\sqrt{2}a)$$

$$= a(a^2-4)$$

Therefore, the determinant of this matrix is zero if $a = 0$, $a = 2$ or $a = -2$.

Problem 7:

$$\begin{aligned} det(2A) &= 2^4 det(A) = 48 \\ det(A^3) &= (det(A))^3 = 27 \\ det(A^{-1}) &= (det(A))^{-1} = \frac{1}{3} \\ det(A^2B^3) &= (det(A))^2(det(B))^3 = 3^2(-2)^3 = -72 \\ det(A^3B^{-2}) &= (det(A))^3(det(B))^{-2} = \frac{27}{4} \end{aligned}$$

Problem 8. Prove that $det(AA^T)$ is nonnegative for any $n \times n$ matrix A.
Proof: $det(AA^T) = det(A)det(A^T) = det(A)det(A) = (det(A))^2 \geq 0$.

Problem 9. Let A be an $n \times n$ matrix and let P be an $n \times n$ invertible matrix. Prove that $det(P^{-1}AP) = det(A)$.
Proof: $det(P^{-1}AP) = det(P^{-1})det(A)det(P) = det(A)(det(P))^{-1}det(P) = det(A)$.

Problem 10. Let A be a $n \times n$ matrix such that $A^T = -A$. Prove that A is not invertible if n is odd.
Proof: $det(A) = det(A^T) = det(-A) = (-1)^n det(A)$. If n is odd, $(-1)^n = -1$. We have, $det(A) = -det(A)$, which leads to $det(A) = 0$. Therefore, A is not invertible.

Problem 11. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Show that i) $A^T = -A$ and ii) A is invertible.
Does this result contradict the conclusion in Problem 10 above?
Proof:

$$A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -A$$

$$det\, A = 0 - (-1) = 1 \neq 0$$

So matrix A is invertible.
This does not contradict the conclusion in Problem 10. In Problem 10, n is odd and here $n = 2$ is even.

Problem 12. We use elementary row operations to calculate these determinants.

$$det \begin{bmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{bmatrix} = det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5$$

$$det \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix} = -det \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} = det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5$$

$$det \left(3 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = 3^3 \cdot det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 3^3 \cdot 5 = 135$$

$$det \begin{bmatrix} a & b & c \\ d & e & f \\ a & b & c \end{bmatrix} = det \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{bmatrix} = 0$$