AMS 10/10A, Homework 6 Solutions

Problem 1: w is in the subspace spanned by $\{v_1, v_2\}$ if w is a linear combination of v_1 and v_2. By elementary row operations, we obtain

$$[v_1\ v_2 \mid w] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, $w = v_1 + 2v_2$, which implies that w is in the subspace spanned by v_1 and v_2.

Problem 2: w is in $Col\ A$ if $Ax = w$ has a solution. By elementary row operations, we obtain

$$[A \mid w] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 3 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The rightmost column is not a pivot column. Therefore, $w \in Col\ A$.

Since $Aw = \begin{bmatrix} 10 \\ -82 \\ 52 \end{bmatrix} \neq 0$, w is not in $Nul\ A$.

Problem 3: $p = 3$ and $q = 5$.

Problem 4: The reduced echelon form of matrix A is

$$A \sim \begin{bmatrix} 1 & -2 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns of A are column 1 and column 3. Therefore, a basis for $Col\ A$ is

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix} \right\}$$

The solution set of $Ax = 0$ in parametric vector form is

$$x = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, a basis for $Nul\ A$ is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Problem 5: Yes. This set of vectors form a basis for $\mathbb{R}^2$, since the set contains 2 linearly independent vectors.

Problem 6: Yes. This set of vectors form a basis for $\mathbb{R}^3$, since the set contains 3 linearly independent vectors.

Problem 7: No. Three linearly independent vectors are needed to form a basis for $\mathbb{R}^3$.

Problem 8: A bass for $Col\ A$ is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 9 \\ 14 \end{bmatrix} \right\}$$

A bass for $Nul\ A$ is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$dim(Col\ A) = 3$, $dim(Nul\ A) = 1$.

Problem 9: A bass for $Col\ A$ is

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ -8 \\ 9 \\ -7 \end{bmatrix} \right\}$$

A bass for $Nul\ A$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$dim(Col\ A) = 2$, $dim(Nul\ A) = 3$.

Problem 10: $dim(Col\ A) = rank = 5, \quad dim(Nul\ A) = 9 - rank = 4$

Problem 11: Mark each statement True or False

11.1. The set of all solutions of a system of homogeneous equation with m equations and n unknowns is a subspace in $\mathbb{R}^m$. **F**

11.2. The set of all linear combinations of columns of an $m \times n$ matrix is a subspace in $\mathbb{R}^n$. **F**

11.3. The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$. **T**

11.4. Let $A = [a_1\ a_2\ a_3]$, where a_1, a_2, and a_3, are vectors in $\mathbb{R}^n$. Then the column space of matrix $[a_1\ a_2\ a_3]$ is the same as the column space of matrix $[a_3\ a_1\ a_2]$. **T**

11.5. The columns of a singular (non-invertible) $n \times n$ matrix may still be a basis for $\mathbb{R}^n$. **F**

Problem 12. Mark each statement True or False

12.1. The dimension of $Col\ A$ is the number of pivot columns in A. **T**

12.2. Suppose A is an invertible $n \times n$ matrix. Then $Col\ A = \mathbb{R}^n$. **T**

12.3. Suppose A is an invertible $n \times n$ matrix. Then $Nul\ A = \{0\}$. **T**

12.4. The dimension of $Nul\ A$ is the number of variables in the equation $Ax = 0$. **F**

12.5. The dimension of $Nul\ A$ is the number of basic variables in the equation $Ax = 0$. **F**

12.6. The dimension of $Nul\ A$ is the number of free variables in the equation $Ax = 0$. **T**

Problem 13. The reduced echelon form of matrix A is

$$A \sim \begin{bmatrix} 1 & 0 & 1.5 & 0 \\ 0 & 1 & -0.5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A bass for $Col\ A$ is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -8 \\ -1 \end{bmatrix} \right\}$$

A bass for $Nul\ A$ is

$$\left\{ \begin{bmatrix} -1.5 \\ 0.5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Problem 14.

1. Yes. $Col\ A = Col\ B$.
2. No. $Nul\ A \neq Nul\ B$. $Nul\ A$ is a subspace in $\mathbb{R}^4$; $Nul\ B$ is a subspace in $\mathbb{R}^5$.