AMS 10/10A, Homework 4 Solutions

Problem 1:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Problem 2: $A(cv) = c(Av) = 0$.

Problem 3: Since v_n is a linear combination of $\{v_1, \cdots, v_{n-1}\}$, there exist scalars c_1, c_2, $\cdots$, c_{n-1} such that

$$v_n = c_1 v_1 + c_2 v_2 + \cdots + c_{n-1} v_{n-1}$$

$$\Longrightarrow c_1 v_1 + \cdots + c_{n-1} v_{n-1} - v_n = 0$$

$$\Longrightarrow \begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} & v_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ 1 \end{bmatrix} = 0 \tag{1}$$

Therefore homogeneous equation $Ax = 0$ has a nontrivial solution which is $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ -1 \end{bmatrix}$. It implies that $Ax = 0$ has infinitely many solutions.

Problem 4: Yes, since A has a pivot position in every row.

Problem 5:

5.1. The equation $Ax = b$ is homogeneous if the zero vector is a solution. **T**

5.2. The homogeneous equation $Ax = 0$ has the trivial solution if and only if the equation has at least one free variable. **F**

5.3. A homogeneous system of equations can be inconsistent. **F**

5.4. If v is a nontrivial solution of $Ax = 0$, then every entry in v is nonzero. **F**

5.5. If homogeneous equation $Ax = 0$ has a unique solution, then $Ax = b$ cannot have infinitely many solutions. **T**

Problem 6:

$$\begin{bmatrix} 2 & 6 & 1 & 5 \\ 1 & 3 & 0 & 2 \\ 3 & 9 & 0 & 6 \\ 1 & 3 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

The set is linearly dependent.

Problem 7:

- Consider vector equation $c_1v_1 + c_2v_2 + c_3v_3 = 0$. It is equivalent to matrix equation

$$\begin{bmatrix} 2 & -1 & 8 \\ -3 & 4 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

  This homogeneous equation has infinitely many solutions given by

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} -6 \\ -4 \\ 1 \end{bmatrix}$$

  Therefore, we have

$$-6v_1 - 4v_2 + v_3 = 0$$

  That is, $\{v_1, v_2, v_3\}$ is linearly dependent.

- $v_2 = -\frac{3}{2}v_1 + \frac{1}{4}v_3$

Problem 8:

$$\begin{bmatrix} 1 & -5 & 3 \\ 3 & -8 & -5 \\ -1 & 2 & k \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & k-3 \end{bmatrix} \tag{3}$$

When $k = 3$ the columns of the matrix are linearly dependent.

Problem 9: Consider vector equation

$$c_1(v_1 + v_3) + c_2(v_1 - 2v_2) + c_3(-4v_1 + v_2 + 3v_3) = 0$$

which can be rewritten as

$$(c_1 + c_2 - 4c_3)v_1 + (-2c_2 + c_3)v_2 + (c_1 + 3c_3)v_3 = 0$$

Since $\{v_1, v_2, v_3\}$ is linearly independent, $\{c_1, c_2, c_3\}$ must satisfy

$$\begin{aligned} c_1 + c_2 - 4c_3 &= 0 \\ -2c_2 + c_3 &= 0 \\ c_1 \qquad + 3c_3 &= 0 \end{aligned} \tag{4}$$

We do row reduction on the coefficient matrix.

$$\begin{bmatrix} 1 & 1 & -4 \\ 0 & -2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -4 \\ 0 & -2 & 1 \\ 0 & 0 & 13/2 \end{bmatrix} \tag{5}$$

Every column is a pivot column. There is no free variable. Equation (4) has only the trivial solution. Therefore, by definition, $\{v_1 + v_3, v_1 - 2v_2, -4v_1 + v_2 + 3v_3\}$ is also linearly independent.

Problem 10: Mark each statement True or False

10.1. The set $\{0, v_1, v_2, \cdots, v_k\}$ is always linearly dependent. **T**

10.2. Let v_1, v_2, v_3 and v_4 be vectors in $\mathbb{R}^n$ such that $v_1 - v_2 = v_3 - v_4$. Then the the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. **T**

10.3. If u and v are linearly independent, and if w is in $span\{u, v\}$, then $\{u, v, w\}$ is linearly dependent.**T**

10.4. If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more than n vectors. **F**

10.5. If $\{v_1, v_2, v_3, v_4\}$ is a set of vectors in $\mathbb{R}^4$ and $\{v_1, v_2, v_3\}$ is linearly dependent, then $\{v_1, v_2, v_3, v_4\}$ is also linearly dependent. **T**

Problem 11:

$$AB = \begin{bmatrix} -3 & 9 \\ 11 & -5 \\ -23 & 6 \end{bmatrix}, \qquad A^T B = \begin{bmatrix} 3 & 5 \\ 36 & -3 \\ -14 & 0 \end{bmatrix}$$

$$BC = \begin{bmatrix} -1 & 4 & -2 & 5 \\ -9 & 1 & 3 & -4 \\ 7 & 7 & -7 & 14 \end{bmatrix} \qquad CD = \begin{bmatrix} -1 & 5 \\ -1 & 2 \end{bmatrix} \qquad (CD)^2 = \begin{bmatrix} -4 & 5 \\ -1 & -1 \end{bmatrix}$$

Problem 12: B has 6 rows.

Problem 13: The second column of AB is a zero column.

Problem 14: When $k = -2$, $AB = BA$.

Problem 15: Let b_1, b_2, $\cdots$, b_n be the columns of B. Since $\{b_1, \cdots, b_n\}$ are linearly dependent, there exist c_1, c_2, $\cdots$, c_n, not all zero, such that

$$\begin{aligned} c_1 b_1 + c_2 b_2 + \cdots + c_n b_n &= 0 \\ \Longrightarrow A\,(c_1 b_1 + c_2 b_2 + \cdots + c_n b_n) &= 0 \\ \Longrightarrow c_1(Ab_1) + c_2(Ab_2) + \cdots + c_n(Ab_n) &= 0 \end{aligned}$$

Therefor, $\{Ab_1, Ab_2, \cdots, Ab_n\}$ are linearly dependent.

Problem 16:

$$\begin{aligned} w_{2,1} &= \begin{bmatrix} 2 & 1 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ 3 & -4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= 6 \end{aligned}$$

Problem 17:

$$\begin{aligned} A^2 - 3A &= \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^2 - \begin{bmatrix} 6 & 3 \\ -3 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 3 \\ -3 & 9 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 \\ -2 & -1 \end{bmatrix} \end{aligned}$$

Problem 18:

$$\begin{aligned} B^3 &= B^2 \cdot B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Therefore, for all $n \geq 3$, $B^n = B^{n-3}B^3 = B^{n-3} \cdot 0 = 0$.