AMS 10/10A, Homework 1 Solutions

For most problems, I only provide the final answers.

Problem 1:

$$\frac{z_1}{z_2} = -\frac{8}{5} - \frac{1}{5}i$$

$$(z_1 + z_2)^3 = -2 - 2i$$

$$(z_1 - \bar{z}_2)^{-2} = \frac{2}{25} + \frac{3}{50}i$$

Problem 2:

$$z = -\frac{8}{13} + \frac{1}{13}i$$

Problem 3:

1. $z_1 = 2e^{\frac{\pi}{6}i}$, $\quad z_2 = \sqrt{2}e^{\frac{5\pi}{4}i}$.
2. $|z_1 \cdot z_2| = 2\sqrt{2}$, $\quad \arg(z_1 \cdot z_2) = \frac{17\pi}{12} + 2k\pi$, $\quad k =$ integer.
3. $|z_1/z_2| = \sqrt{2}$, $\quad \arg(z_1/z_2) = \frac{11\pi}{12} + 2k\pi$, $\quad k =$ integer.

Problem 4:

1. $\cos\alpha - i\sin\alpha = \cos(-\alpha) + i\sin(-\alpha) = e^{-\alpha i}$;
2. $\sin\alpha + i\cos\alpha = \cos(\frac{\pi}{2} - \alpha) + i\sin(\frac{\pi}{2} - \alpha) = e^{(\frac{\pi}{2}-\alpha)i}$;
3. $\sin\alpha - i\cos\alpha = \cos(\alpha - \frac{\pi}{2}) + i\sin(\alpha - \frac{\pi}{2}) = e^{(\alpha-\frac{\pi}{2})i}$.

Problem 5:

1. $x = 0$ and $x = 2^{1/3}$;
2. $x = 0$, $\quad x = 2^{1/3}$, $\quad x = 2^{1/3}e^{\frac{2\pi}{3}i}$ and $x = 2^{1/3}e^{\frac{4\pi}{3}i}$

Problem 6:

$$-1 + i = \sqrt{2}e^{\frac{3\pi}{4}i}$$

The solutions of $z^3 = -1 + i$ are

$$z = 2^{\frac{1}{6}} e^{\frac{\pi}{4} i}, \quad z = 2^{\frac{1}{6}} e^{\frac{11\pi}{12} i}, \quad z = 2^{\frac{1}{6}} e^{\frac{19\pi}{12} i}$$

Problem 7:

$$i = e^{\frac{\pi}{2} i}$$

The solutions of $z^4 = i$ are

$$z = e^{\frac{\pi}{8} i}, \quad z = e^{\frac{5\pi}{8} i}, \quad z = e^{\frac{9\pi}{8} i}, \quad z = e^{\frac{13\pi}{8} i}$$

Problem 8:

$$-1 + i\sqrt{3} = 2e^{\frac{2\pi}{3} i}$$

The solutions of $z^4 = -1 + i\sqrt{3}$ are

$$z = 2^{\frac{1}{4}} e^{\frac{\pi}{6} i}, \quad z = 2^{\frac{1}{4}} e^{\frac{2\pi}{3} i}, \quad z = 2^{\frac{1}{4}} e^{\frac{7\pi}{6} i}, \quad z = 2^{\frac{1}{4}} e^{\frac{5\pi}{3} i}$$

Problem 9:

1. $\sqrt{3} - i = 2e^{\frac{-1\pi}{6} i}$;
2. $z^{13} = 2^{13} e^{\frac{-13\pi}{6} i} = 2^{13} e^{\frac{-1\pi}{6} i} = 2^{12}\sqrt{3} - 2^{12}\ i$
3. $z^{22} = 2^{22} e^{\frac{-22\pi}{6} i} = 2^{22} e^{\frac{2\pi}{6} i} = 2^{21} + 2^{21}\sqrt{3}\ i$

Problem 10: Let $z = a + bi$.

$$3^2 = |z - (1+i)|^2 \quad = \quad |(a-1) + (b-1)i|^2 \;= (a-1)^2 + (b-1)^2$$

Therefore, $|z - (1+i)| = 3$ describes a circle of radius 3, centered at $(1, 1)$.

Problem 11: Let $z = re^{\theta i}$.

$$\sqrt{2} + i\sqrt{2} = 2e^{\frac{\pi}{4} i}$$

$$z \times (\sqrt{2} + i\sqrt{2}) = (2r) e^{(\theta + \frac{\pi}{4}) i}$$

Problem 12: For a polynomial of real coefficients, we have the decomposition

$$P(z) = a_n z^n + \cdots + a_1 z + a_0 = a_n (z - \xi_1) \cdots (z - \xi_m) Q_1(z) Q_2(z) \cdots Q_k(z)$$

where $\xi_1, \xi_2, \ldots, \xi_k$ are the real roots of $P(z)$, and each $Q_j(z)$ is a quadratic polynomial corresponding to a conjugate pair of complex roots $(\zeta_j, \overline{\zeta_j})$

$$Q_j(z) = z^2 - Re(\zeta_j) + |\zeta_j|^2$$

If there is no real root, the right hand side is a product of quadratic polynomials $Q_j(z)$'s. As a result, the degree of the right hand side is even, which contradicts with the given condition that the polynomial is of odd degree. Therefore, a polynomial of odd degree with real coefficients must have at least one real root.