AMS 10/10A, Homework 10 Solutions

Problem 1.

$$\begin{aligned}\hat{y} &= \frac{y^T u_1}{u_1^T u_1} u_1 + \frac{y^T u_2}{u_2^T u_2} u_2 \\ &= \frac{1}{3} u_1 + \frac{5}{7} u_2 \\ &= \begin{bmatrix} -8/21 \\ 52/21 \\ -23/21 \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}z &= y - \hat{y} \\ &= \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} -8/21 \\ 52/21 \\ -23/21 \end{bmatrix} \\ &= \begin{bmatrix} 50/21 \\ -10/21 \\ -40/21 \end{bmatrix}\end{aligned}$$

They satisfy that $y = \hat{y} + z$, where $\hat{y}$ is a vector in H and z is a vector in $H^{\perp}$.

Problem 2. By Best Approximation Theorem the closest point in $span\{v_1, v_2\}$ to y is given by the projection of y onto $span\{v_1, v_2\}$. Since v_1 and v_2 are orthogonal, this projection can be computed as

$$\frac{y^T v_1}{v_1^T v_1} v_1 + \frac{y^T v_2}{v_2^T v_2} v_2 = \frac{1}{5} v_1 + \frac{1}{13} v_2 = \begin{bmatrix} -7/65 \\ -21/65 \\ -1/5 \\ 41/65 \end{bmatrix}$$

Problem 3-7.

- By applying elementary row operations on the augmented matrix $[A \,|\, b]$, we have

$$[A \,|\, b] \sim \left[\begin{array}{cc|c} 1 & 5 & 1 \\ 0 & -14 & -3 \\ 0 & 0 & 4 \end{array}\right]$$

  Since the last column is a pivot column, the equation $Ax = b$ is inconsistent.

- Since the columns of A are an orthogonal set of non-zero vectors, they are a linearly independent set. Consequently, they form an orthogonal basis for $Col(A)$.

- The column of A are an orthogonal basis for $Col(A)$. Hence, projection of b onto $Col(A)$ is given by

$$\hat{b} = \frac{b^T a_1}{a_1^T a_1} a_1 + \frac{b^T a_2}{a_2^T a_2} a_2 = \frac{1}{2} a_1 - \frac{1}{6} a_2 = \begin{bmatrix} -1/3 \\ 4/3 \\ 5/3 \end{bmatrix}$$

- The least square solution, $\hat{x}$, of $Ax = b$ is given by

$$\begin{aligned} \hat{x} &= (A^T A)^{-1} A^T b \\ &= \begin{bmatrix} 14 & 0 \\ 0 & 42 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ -7 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/6 \end{bmatrix} \end{aligned}$$

- $A\hat{x} = \begin{bmatrix} -1/3 \\ 4/3 \\ 5/3 \end{bmatrix} = \hat{b}.$

Problem 8-9. Let A be an $m \times n$ matrix. Use the steps below to show that a vector x in $\mathbb{R}^n$ satisfies $Ax = 0$ if and only if $A^T A x = 0$.

- Show that if $Ax = 0$, then $A^T A x = 0$.

  Proof: Let x be a vector such that $Ax = 0$. Multipling A^T on both sides of the equation leads to $A^T A x = A^T 0 = 0$.

- Suppose $A^T A x = 0$. Show that $x^T A^T A x = 0$, and use this to prove $Ax = 0$.

  Proof: Let x be a vector such that $A^T A x = 0$. Multipling x^T on both sides of the equation leads to $x^T A^T A x = x^T 0 = 0$. Therefore, $x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 = 0$. Since the norm of a vector equals to zero if and only if the vector itself is the zero vector, we have $Ax = 0$.

Problem 10-11. Let A be an $m \times n$ matrix. Problem 8-9 implies that $Nul(A) = Nul(A^T A)$. Use this result to prove that

- $rank(A) = rank(A^T A)$.

  Proof: Matrix A is $m \times n$ and matrix $A^T A$ is $n \times n$. By the Rank Theorem, we have

$$rank(A) + dim(Nul\ A) = n$$
$$rank(A^T A) + dim(Nul\ A^T A) = n$$

  Hence,

$$\begin{aligned} rank(A) &= n - dim(Nul\ A) \\ &= n - dim(Nul(A^T A)) \qquad \text{(since } Nul(A) = Nul(A^T A)\text{)} \\ &= rank(A^T A) \qquad \text{(By The Rank Theorem)} \end{aligned}$$

- If $rank(A) = n$, then $A^T A$ is invertible.

  Proof: If $rank(A) = n$, from the result in Problem 10, $rank(A^T A) = rank(A) = n$. Since matrix $A^T A$ is a square matrix of $n \times n$, by Invertible Matrix Theorem, $rank(A^T A) = n$ implies that $A^T A$ is invertible.