Lower bounds for Sorting

A bound on the complexity of a problem is a bound on ***any*** algorithm that solves the problem

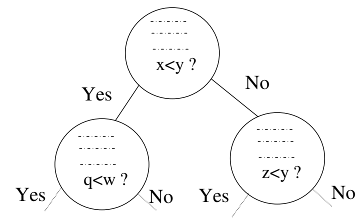
Techniques:

- Decision Tree Model
  - for comparison based algorithms (i.e. sorting)
- Adversary Argument
  - for selection problems
- Reduction
  - show a problem solves another problem with a known lower bound

Decision Tree Model of Computation

Outcomes of comparisons determine algorithm branching

Keys are only used in comparisons (no indexing)



If by varying keys *m* outcomes are possible
then the Decision Tree must have at least *m* leaves

Decision Tree Model of Computation (cont)

Any comparison-based sorting algorithm requires

$\Omega(n \lg n)$ ***comparisons in the worst-case.***

Proof: Let ***A*** be any comparison-based sorting algorithm.

Let *T* be the decision tree corresponding to ***A***.

There are *n!* outcomes of sorting *n* keys.

T must have at least *n!* leaves.

A tree with *k* leaves has height at least lg *k*.

T has height at least $\lg n! = \Theta(n \lg n)$.

A makes at least $\Omega(n \lg n)$ comparisons in the worst-case.

QED

Decision Tree Model of Computation (cont)

Any comparison-based sorting algorithm requires

$\Omega(n \lg n)$ ***comparisons on average.***

Proof: Let ***A*** be any comparison-based sorting algorithm.

Let *T* be the decision tree corresponding to ***A***.

There are *n!* outcomes of sorting *n* keys.

Assume all *n!* outcomes are equally probable.

Expected number of comparisons is:

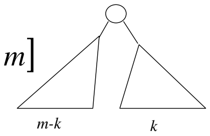
$$\sum_{l \in L(T)} \frac{1}{n!} \cdot depth(l) = \frac{1}{n!} \cdot (\text{external path length of } T)$$

Decision Tree Model of Computation (cont)

Proof: (continued)

Let $D(m)$= minimum external path length of any tree with m leaves.

$$D(m) = \min_{1 \le k < m} \left[D(m-k) + D(k) + m \right]$$



Show $D(m) \ge m \lg m$ by substitution method.

$$\sum_{l \in L(T)} \frac{1}{n!} \cdot depth(l) \ge \frac{1}{n!} \cdot (n! \lg n!) = \lg n! = \Theta(n \lg n)$$

Expected number of comparisons is $\Omega(n \lg n)$.

QED