Example big-O proof

Here we prove that $10(\lg n)^3 \in O(n^{1/5})$. The definition of big-$O$ states that we must show, for some positive c and n_0, the following

$$0 \leq 10(\lg n)^3 \leq c(n^{1/5}) \text{ for all } n \geq n_0.$$

Theorem: $10(\lg n)^3 \in O(n^{1/5})$.
Proof: We will use $c = 10$ and $n_0 = 2^{150}$.
Start with the fact:

$$0 \leq \lg(2^{150}) = 150 < 1024 = (2^{150})^{1/15}.$$

Note that the derivative of $\lg n$ is $\frac{1}{n}$ while the derivative of $n^{1/15}$ is $\dfrac{n^{1/15}}{n}$. So when $n > 1$ (and thus for $n > 2^{150}$), the function $\lg n$ grows more slowly than $n^{1/15}$. Since $\lg n$ is less than $n^{1/15}$ when $n = 2^{150}$ (by the above fact), $\lg n$ is less than $n^{1/15}$ for all $n \geq 2^{150}$.
Therefore:

$$\begin{aligned}
0 \leq \lg n \leq n^{1/15} &\quad \text{for all } n \geq 2^{150} \\
0 \leq (\lg n)^3 \leq n^{1/5} &\quad \text{for all } n \geq 2^{150} \\
0 \leq 10(\lg n)^3 \leq 10n^{1/5} &\quad \text{for all } n \geq 2^{150} \\
0 \leq 10(\lg n)^3 \leq cn^{1/5} &\quad \text{for all } n \geq n_0
\end{aligned}$$

for $c = 10$ and $n_0 = 2^{150}$. So $10(\lg n)^3 \in O(n^{1/5})$ by the definition of big-O.