CMPE-242
Applied Feedback Control

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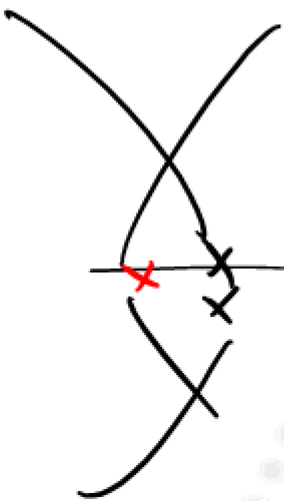
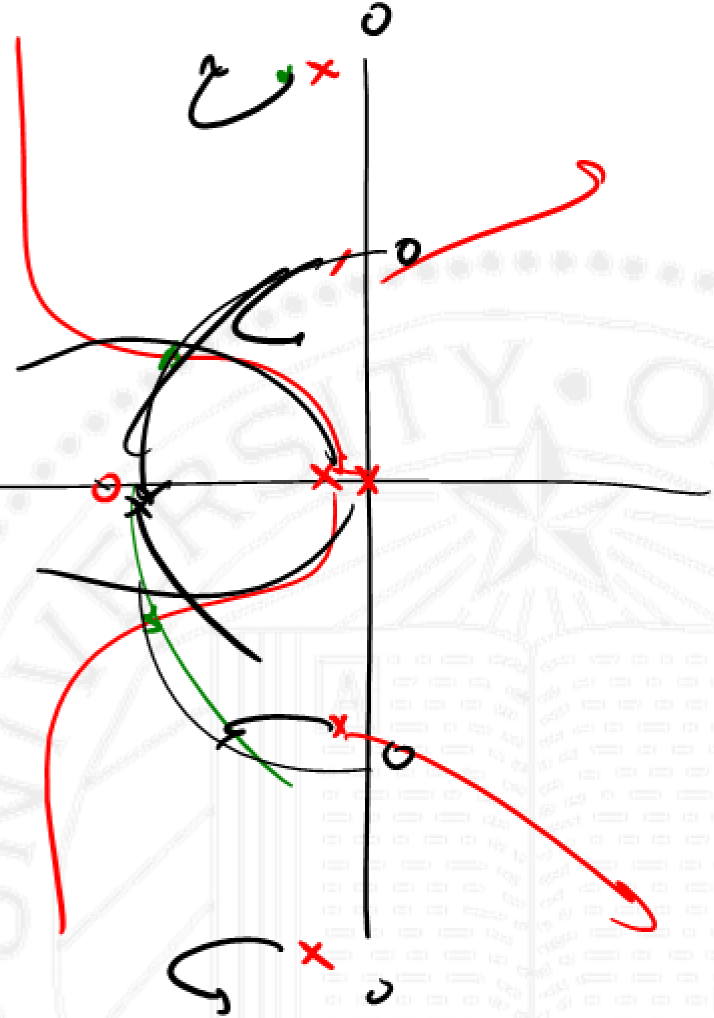
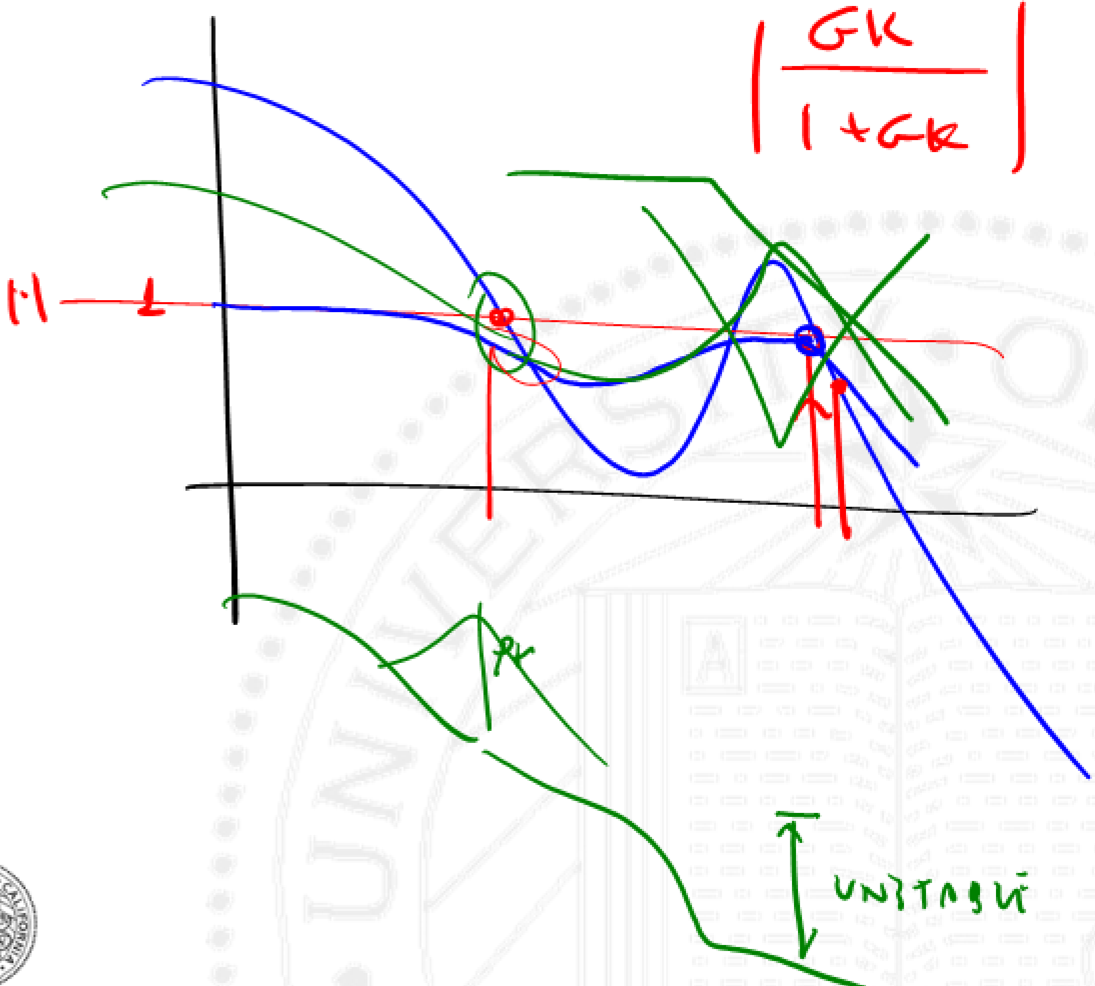
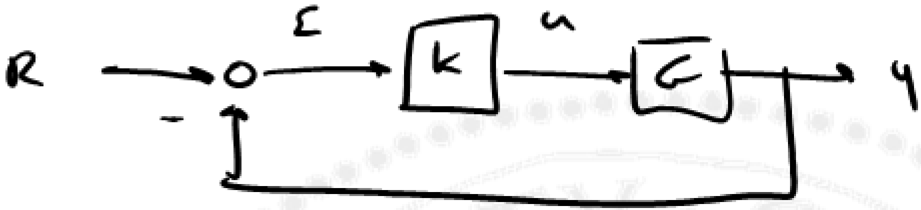
$2\pi = \omega_{xo}$






$\left|\frac{GK}{1+GK}\right|$
1

u, θ



$$\frac{y}{R} = \frac{GK}{1+GK}$$

$$\frac{U}{R} = \frac{K}{1+GK}$$

$$\dot{\hat{x}} = \underbrace{[A - LC]}_{\text{compute offline + store}}\hat{x} + \underbrace{[B \,\vdots\, L]}_{\text{Doesn't change.}}\begin{bmatrix} u \\ y \end{bmatrix}$$

$u = -K\hat{x}$ choose $K = \text{place}(A, B, P_{des})$

① full rank / cond.

$L^T = \text{place}(A^T, C^T, P_{est})$

② full rank / cond.

$|P_{est}| > 5\text{-}10 \times |P_{cont}|$

$$G(s) = \frac{Y}{U} = \frac{1}{s^2} = \frac{1}{s^2 + 0s + 0}$$

$$s_{des} = -1 \pm j$$

$$\Delta_d = s^2 + 2s + 2$$

$$\underline{x} = \begin{bmatrix} \dot{y} \\ y \end{bmatrix}$$

$$\begin{bmatrix} \ddot{y} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$\uparrow$

$\underline{\dot{x}}$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{y} \\ y \end{bmatrix} + [0] u$$

$$\mathcal{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\text{rank}(\mathcal{C}) = 2$

$\text{cond} = 1.$

$$\mathcal{O} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\text{rank}(\mathcal{O}) = 2$

$\text{cond}(\mathcal{O}) = 1.$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$P_{des} = -1 \pm j \qquad K = \text{place}(A, B, P_{des}) \Rightarrow K = [2\ 2]$

$$A - BK = \begin{bmatrix} -k_1 & -k_2 \\ 1 & 0 \end{bmatrix} \qquad \det(sI - A + BK) = \det\begin{pmatrix} s+k_1 & k_2 \\ -1 & s \end{pmatrix}$$

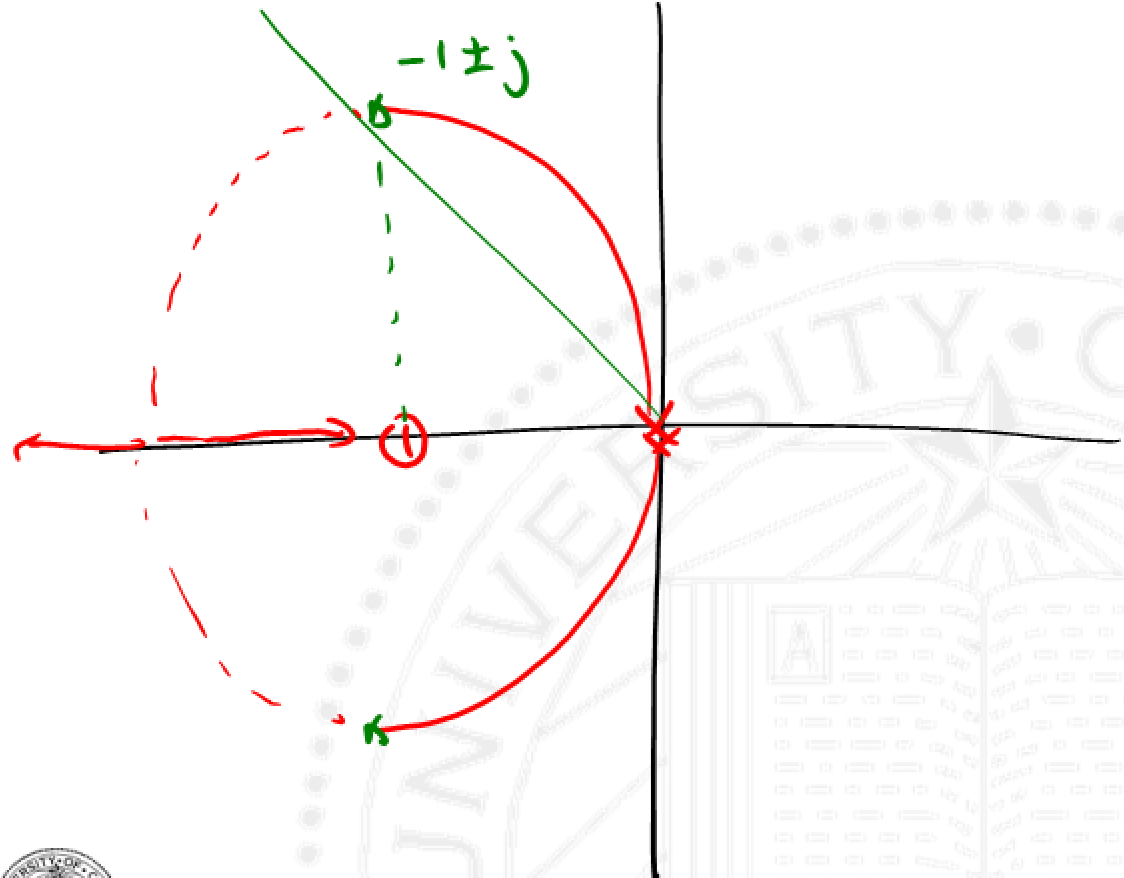
$$s(s+k_1) + k_2 = 0 \qquad s^2 + \boxed{k_1} s + \boxed{k_2} = 0$$

$$s^2 + \boxed{2} s + \boxed{2} = 0$$

$$u = -kx \Rightarrow -2\dot{y} - 2y = u \qquad u = -2(s+1)y$$

$$\frac{u}{y} = -2(s+1)$$

has zero.



$-1 \pm j$

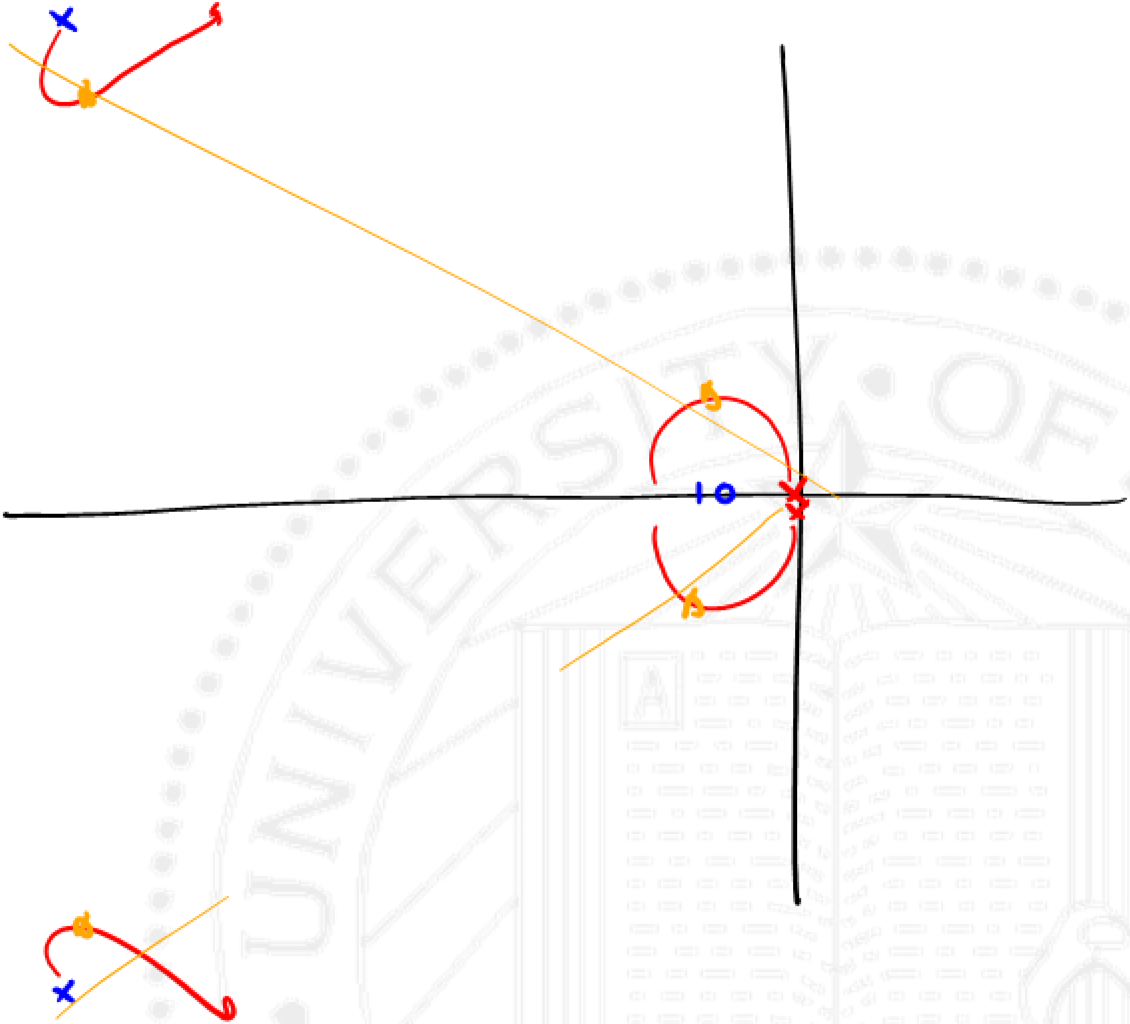
$$P_{est} = -10 \pm 10j \qquad L^T = \text{place}(A^T, C^T, P_{est}) = \begin{bmatrix} 200 \\ 20 \end{bmatrix}$$

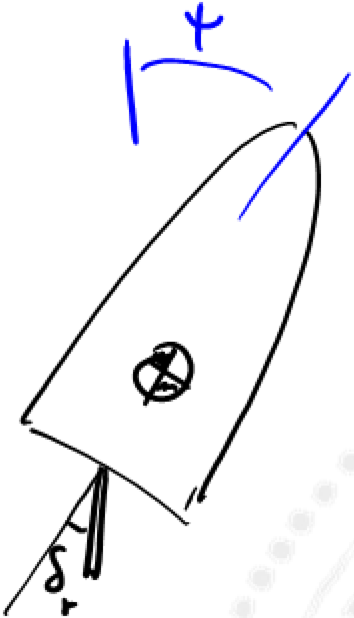
$$\det(sI - A + LC) = \Delta_{d_{est}} = s^2 + 20s + 200$$

$$\left.\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \overset{C\hat{x}}{\hat{y}}) \\ u &= -K\hat{x} \end{aligned}\right\} \rightarrow \begin{aligned} \dot{\hat{x}} &= (A - BK - LC)\hat{x} + Ly \\ u &= -K\hat{x} \end{aligned}$$

$$K(s) = -K\left[sI - A + BK + LC\right]^{-1} L$$

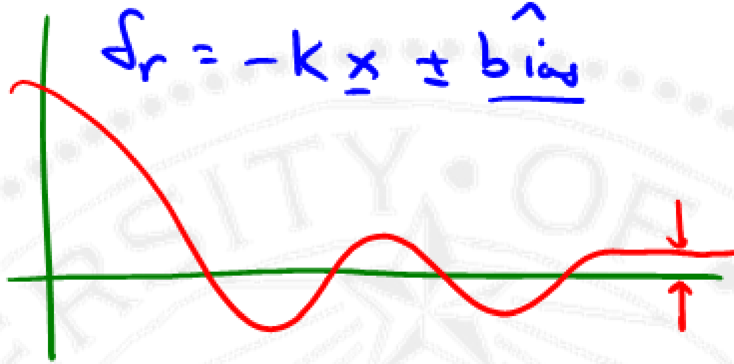
$$= \boxed{\frac{940\,(s + 0.91)}{s^2 + 22s + 262}}$$





$\delta_{r_{true}} = \delta_{r_{meas}} + bias$

$\delta_r = -K\underline{x} \pm \underline{\hat{bias}}$



$\underline{x} = \begin{bmatrix} \psi \\ \dot{\psi} \\ \delta_r \\ b_r \end{bmatrix} \longrightarrow \dot{\underline{x}} = \left[\begin{array}{c|c} A & \begin{matrix} B \\ -1 \end{matrix} \\ \hline 0 - 0 & 0 \end{array}\right]$

$\delta_r = \delta_{r_m} - b_r$

$\mathcal{O} \stackrel{?}{=}$ full rank

Control

$$\dot{x} = Ax + Bu$$

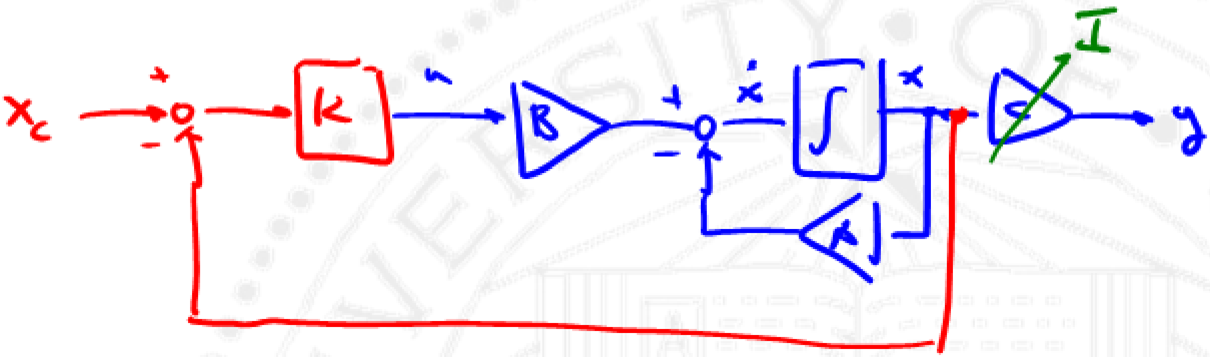
$$y = Cx + Du$$

commanded state

$$u = -K(x - x_c)$$

means full state

$\mathcal{C}$ full rank.



Form $\mathcal{C} = \left[B \;\; AB \;\; A^2B \; \ldots \; A^{n-1}B \right]$ rank$(\mathcal{C}) = n$

rcond$(\mathcal{C}) = 1 \longrightarrow 0$

put poles of $A - BK$ anywhere I want

Kalman Filter

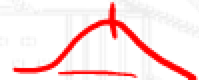
$$\dot{x} = Ax + Bu + \Gamma w$$ process noise

$$y = Cx + v$$ measurement noise

$$w = N(0, R_w)$$ [ ]

$$v = N(0, R_v)$$ [ ]

GAUSSIAN DISTRIBUTION



BLUE – BEST LINEAR UNBIASED ESTIMATOR

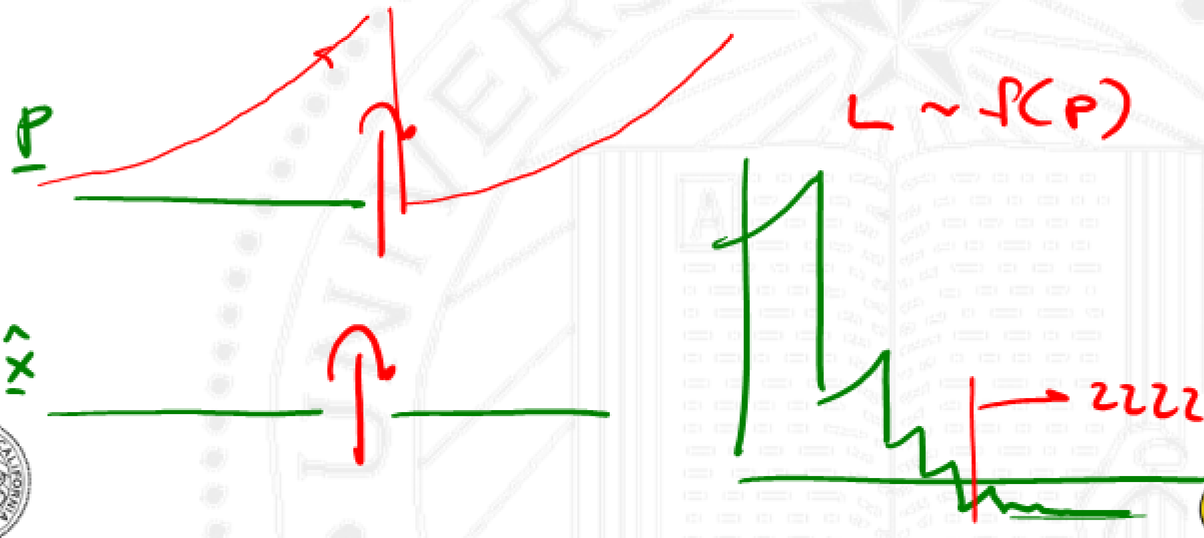
small L ← trust my model, not measurements.

big L ← trust my measurements, ignore model

Kalman Filter – lqe linear quadratic estimator

P ← covariance matrix $E(\tilde{x}^T \tilde{x})$

True Kalman Filter — continuously variable linear est.



F.V.T.

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$\Rightarrow$ FVT $\rightarrow \dot{x} = 0$.

$$\overset{n\times 1}{0} = Ax_{ss} + Bu_{ss} \qquad x_{ss} = -A^{-1}Bu_{ss}.$$

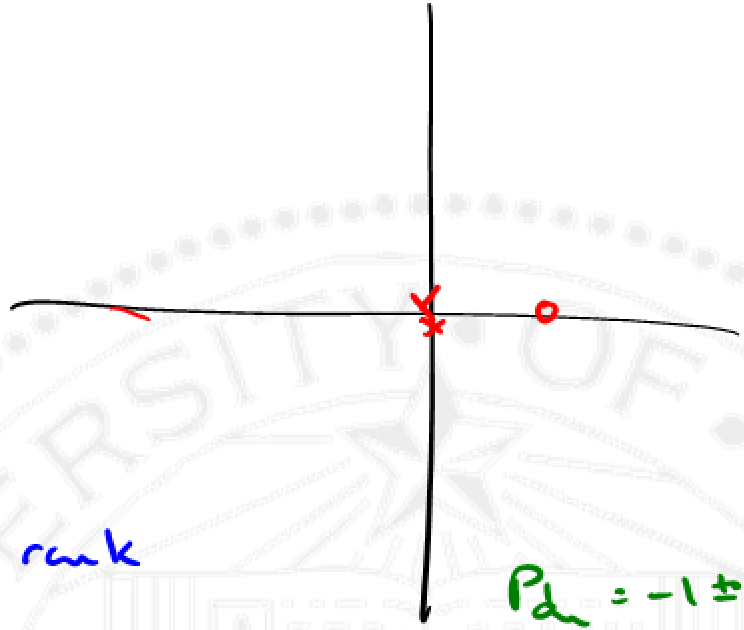
$$y_{ss} = Cx_{ss} + Du_{ss} = \left[C\left(-A^{-1}B\right) + D\right]u_{ss}$$

$$\frac{y_{ss}}{u_{ss}} = \underbrace{\left[-CA^{-1}B + D\right]}_{\text{DC gain matrix}}$$

$$G(s) = -\frac{(s-1)}{s^2}$$

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$



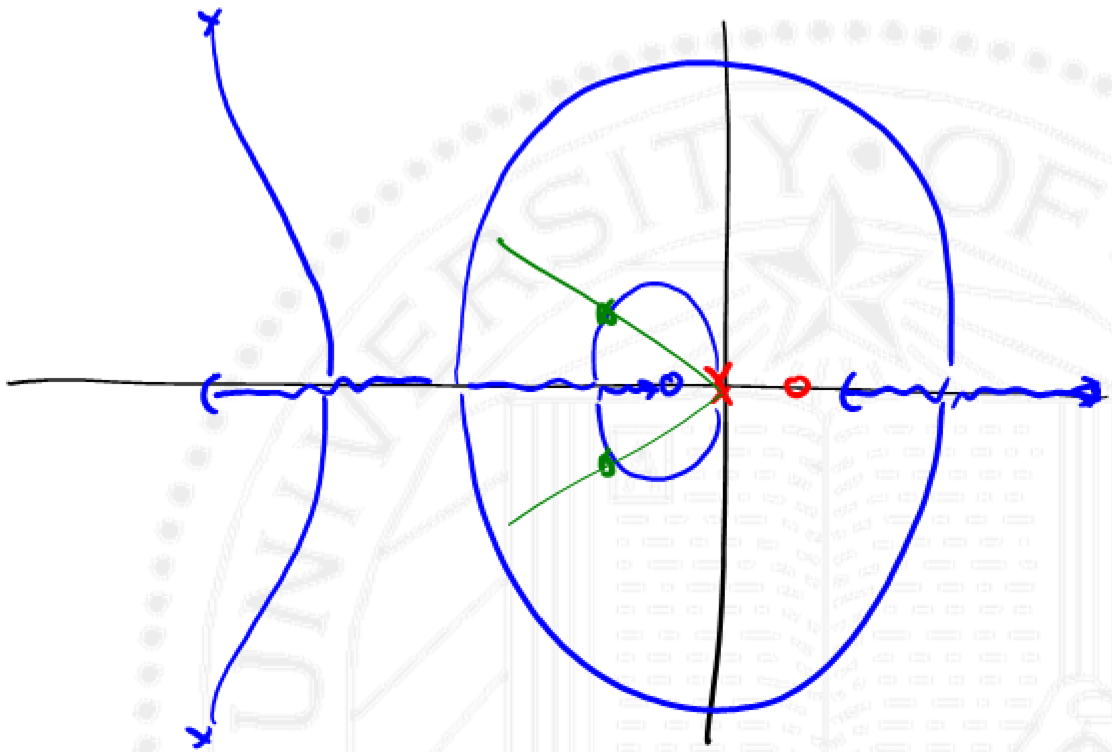
$\mathcal{C}$ = unchanged, full rank

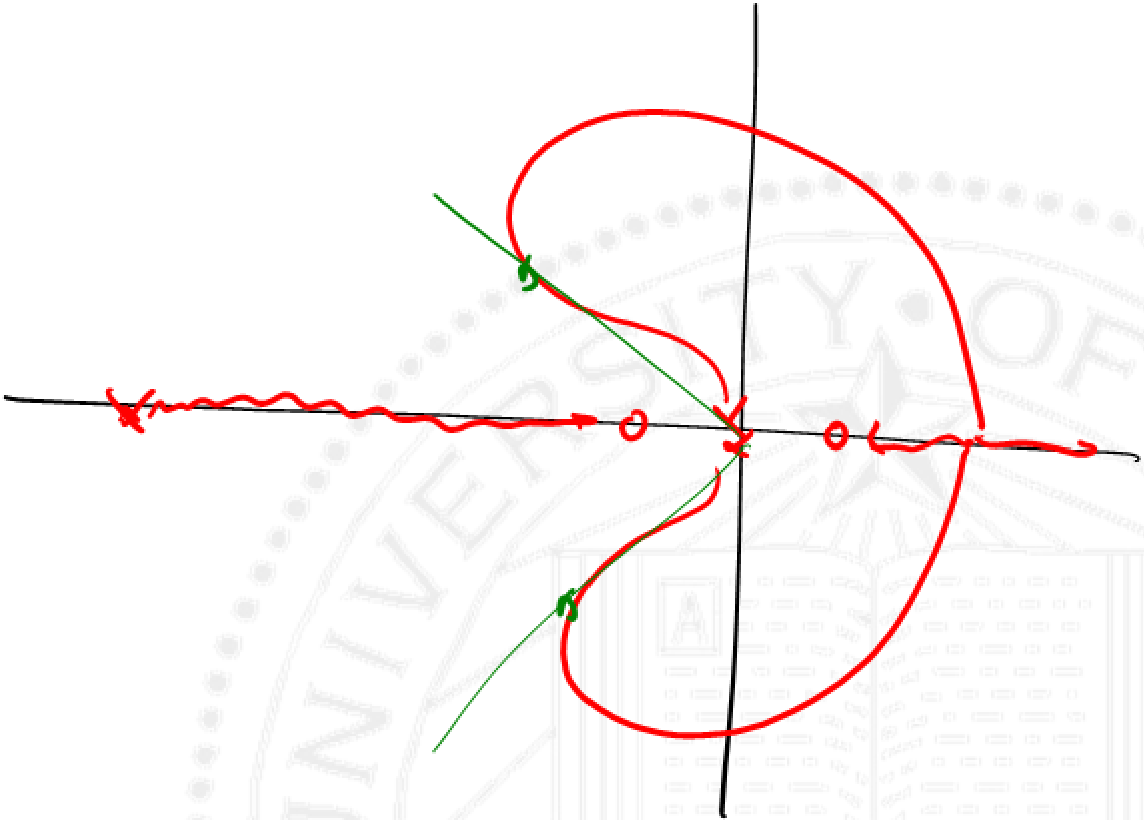
$$P_d = -1 \pm j$$

$$P_{obs} = -10 \pm 10j$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$ full rank.

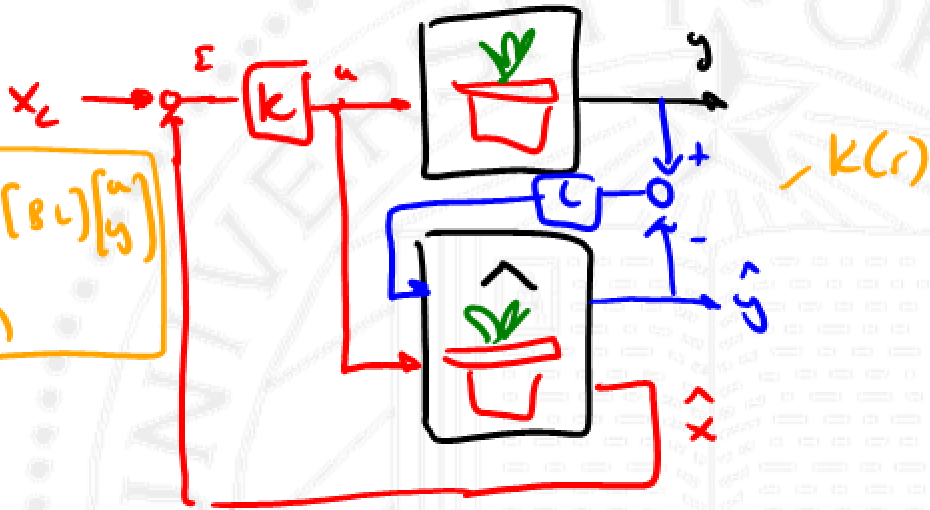
$$K(s) = \frac{840\,(s+0.476)}{s^2 + 22s + 1082} \leftarrow -11 \pm 31j$$





$K = \text{place}(A, B, P_{control})$ "$A - BK$"

$L^T = \text{place}(A^T, C^T, P_{est})$ "$A - LC$"



$$\dot{\hat{x}} = [A - LC]\hat{x} + [B\ L]\begin{bmatrix} u \\ y \end{bmatrix}$$

$$u = -K(\hat{x} - x_r)$$

$$\frac{U}{Y} = -K[sI - A + LC + BK]^{-1}L \leftarrow \text{compensator}$$

Ways to choose L

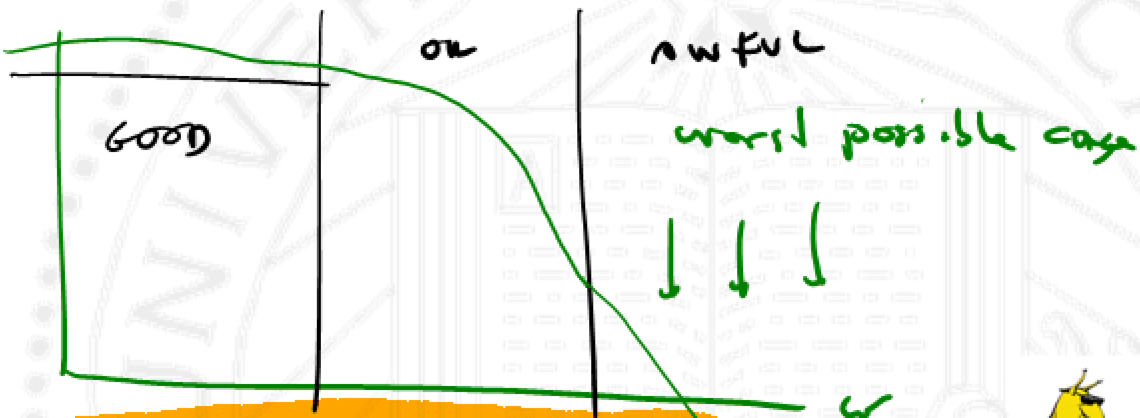
$L^T = \text{place}(A^T, C^T, P_{esd})$ $P_{esd} > 5\times10 \times P_{ctrl}$

Kalman Filter ~ R_w, R_v lqe.

Better Ways to Choose Pole Locations

H_2 — lqr — min $\| \ \|_2$ of x^Tx u^Tu

H_∞ " $\| \ \|_\infty$ of x^Tx u^Tu



ALWAYS PLOT CONTROL EFFORT

$$\dot{x} = Ax + Bu \qquad u = -K(x - x_c) = \boxed{Kx_c - Kx}$$

$$\dot{x} = (A - BK)x + BK\,\underline{x_c}$$

$$y = (C - DK)\underline{x} + DK\,\underline{x_c}$$

$$\begin{bmatrix} y \\ \cdots \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} C - DK \\ \cdots\cdots \\ -K \end{bmatrix}}_{\tilde{C}} \underline{x} + \underbrace{\begin{bmatrix} DK \\ \cdots \\ K \end{bmatrix}}_{\tilde{D}} \underline{x_c}$$

LQR control – LINEAR QUADRATIC REGULATOR

$$\min_{K} \; J = \int_0^{\infty} \left(x^T Q x + u^T R u\right) dt$$

s.t.:
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

ALGEBRAIC RICCATI EQUATION

$$\left[A^T P + P^T A + P^T B R^{-1} B^T P + Q = 0\right] \leftarrow \text{solve for } P.$$

$$u = -\underbrace{R^{-1} B P}_{K} x$$

$$\boxed{u = -Kx}$$

$K = \mathrm{lqr}(A, B, Q, R)$

"TUNING KNOBS"

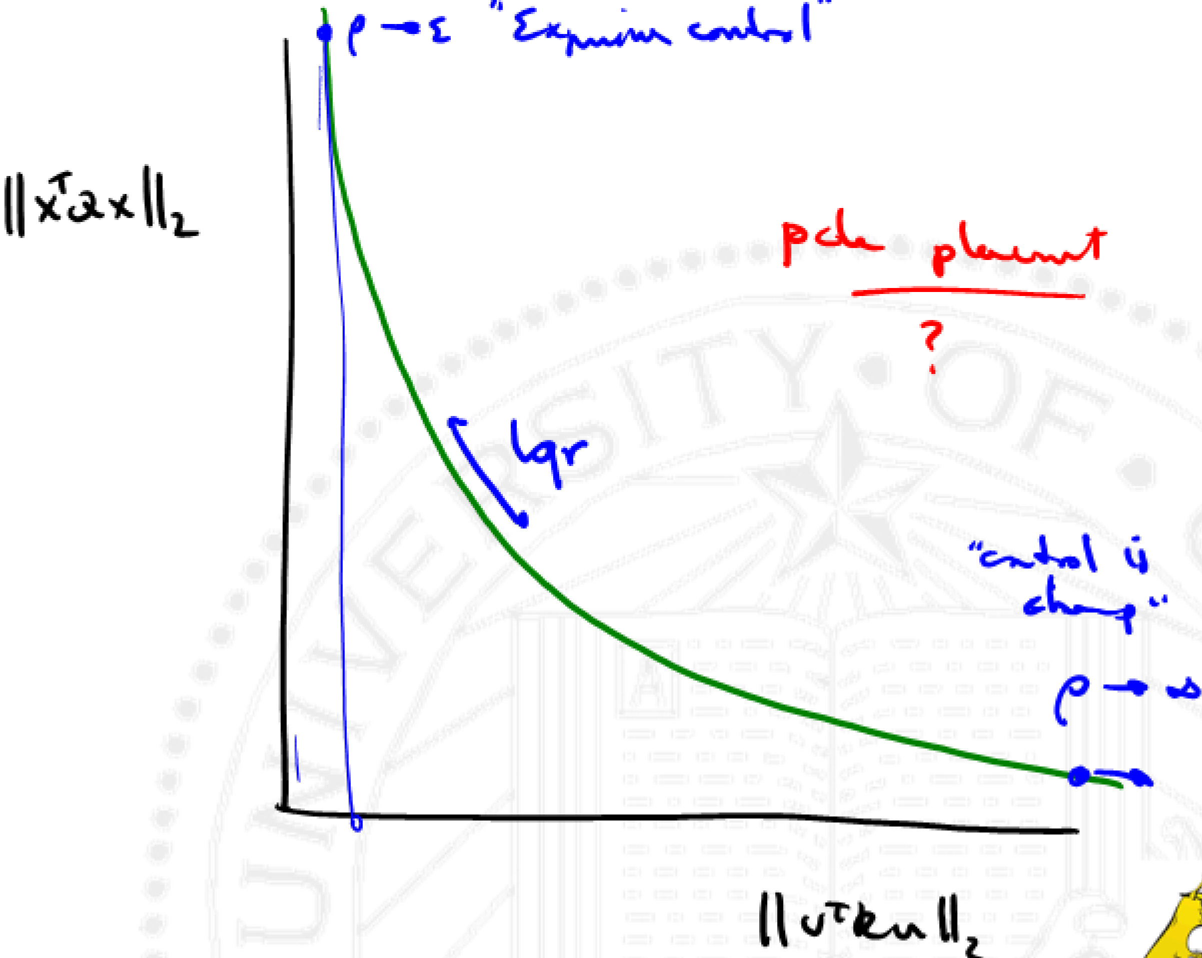
$\rho \to \varepsilon$: huge control effort.

$Q, R \leftarrow$ Bryson's Rule

$$Q = \frac{1}{\rho}\begin{bmatrix} 1/x_{1,max}^2 & & 0 \\ & \ddots & \\ 0 & & 1/x_{n,max}^2 \end{bmatrix} \qquad R = \begin{bmatrix} 1/u_{1,max}^2 & & \\ & \ddots & \\ & & \frac{1}{u_{m,max}^2} \end{bmatrix}$$

$$J = \int_0^{\infty} \rho \left[\left(\frac{x_1}{x_{1,max}}\right)^2 + \dots + \left(\frac{x_n}{x_{n,max}}\right)^2 \right] + \left[\left(\frac{u_1}{u_{1,max}}\right)^2 + \dots + \left(\frac{u_m}{u_{m,max}}\right)^2 \right] dt$$

tune ρ .


$\rho \to \varepsilon$ "Expensive control"
$\|x^TQx\|_2$
pole placement
?
lqr
"control is cheap"
$\rho \to \infty$
$\|u^TRu\|_2$

LQR – full state feedback

C full rank

design knob.

Bryson's Rule for Q, R tune w/ ρ

$$\boxed{PM \geq 50^\circ \qquad \frac{1}{2} < GM < 2}$$

$y = Cx \qquad y^T = x^T C^T$

$$J = \int_0^\infty \left(\rho\, y^T \underbrace{Q_y}_{I}\, y + u^T R u\right) dt$$

$$J = \int_0^\infty \rho\, x^T \underbrace{C^T Q_y C}_{Q} x + \dots$$

$$Q_y = \begin{bmatrix} \frac{1}{y_{1,max}^2} & & \\ & \ddots & \\ & & \frac{1}{y_{r,max}^2} \end{bmatrix}$$

Optimal Control — LQ "R" regulator

"E" estimator/Kalman

"G" gaussian

$K = lqr(A, B, Q, R)$

$K = lqy(A, B, Q_y, R)$

$L = lqe(A, B, R_w, R_v)$

LQR : 50° PM

$\frac{1}{2}$ GM $\in (2, \infty)$

John Doyle : "Performance Guarantees of LQR control"

Abstract : There are none.

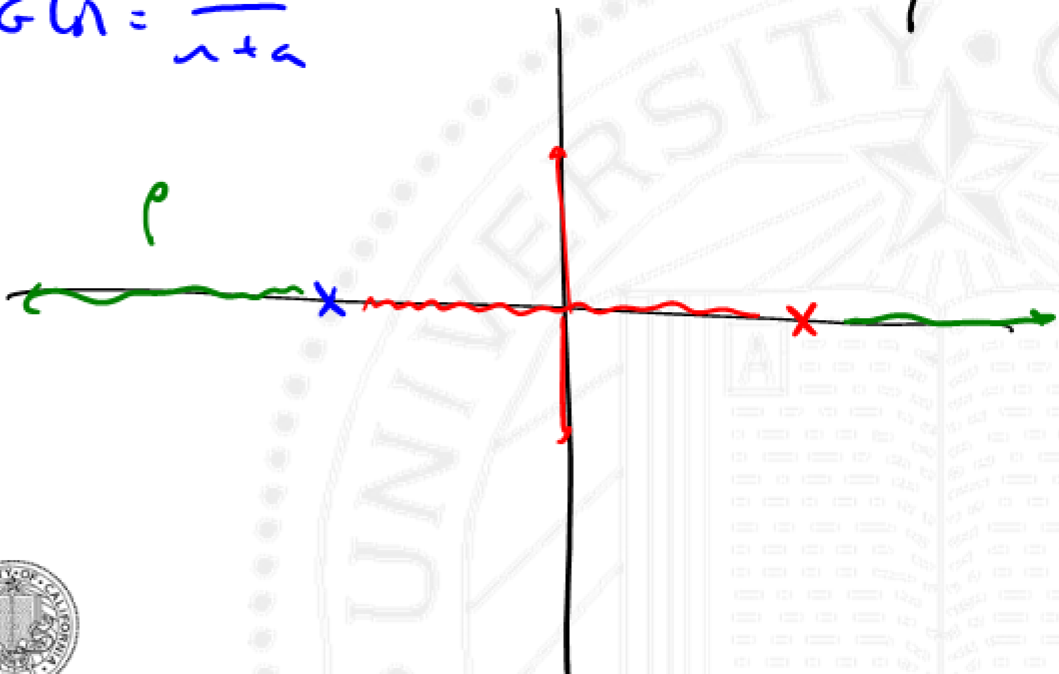
SISO

$$J = \int_0^\infty (\rho y^2 + u^2)\, dt$$

Solution is symmetric root locus

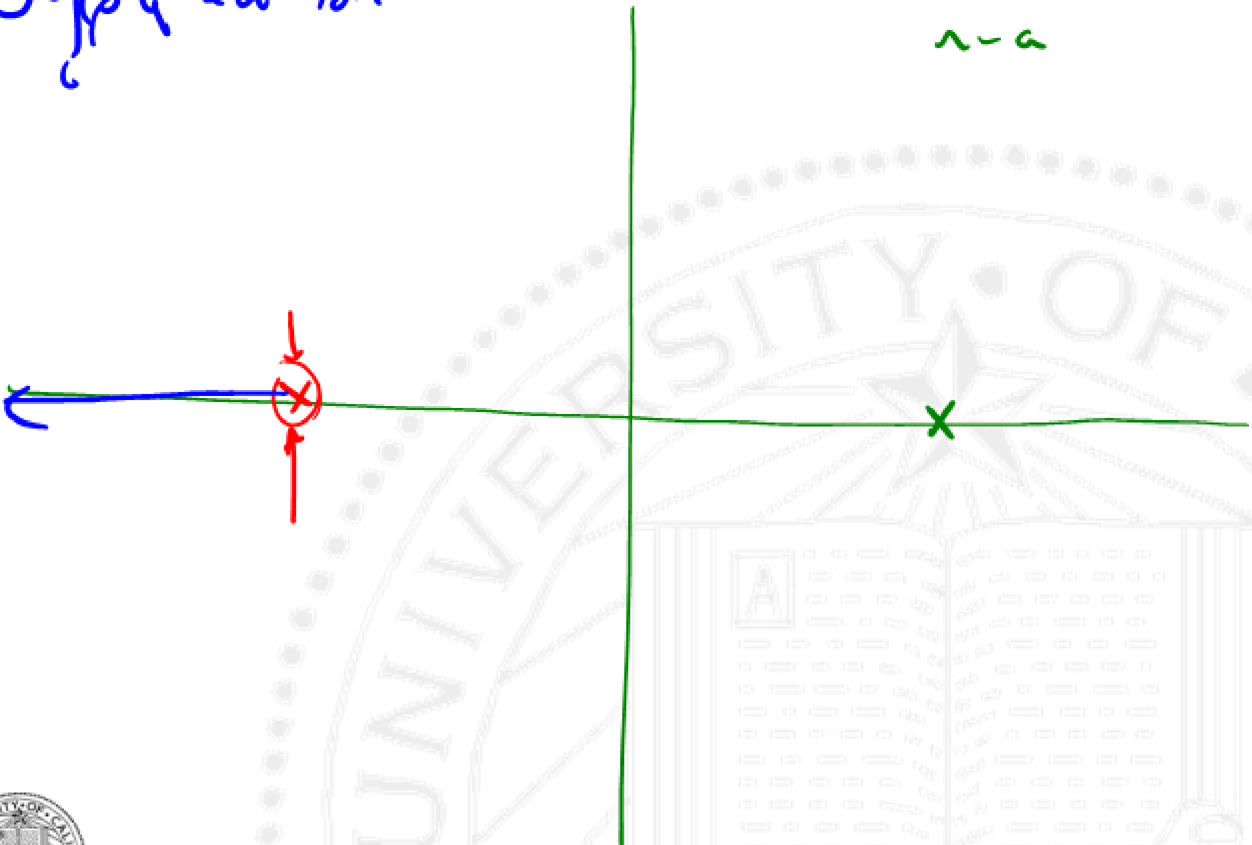
$$1 + \rho G(s) G(-s) = 0.$$

$$G(s) = \frac{1}{s+a}$$

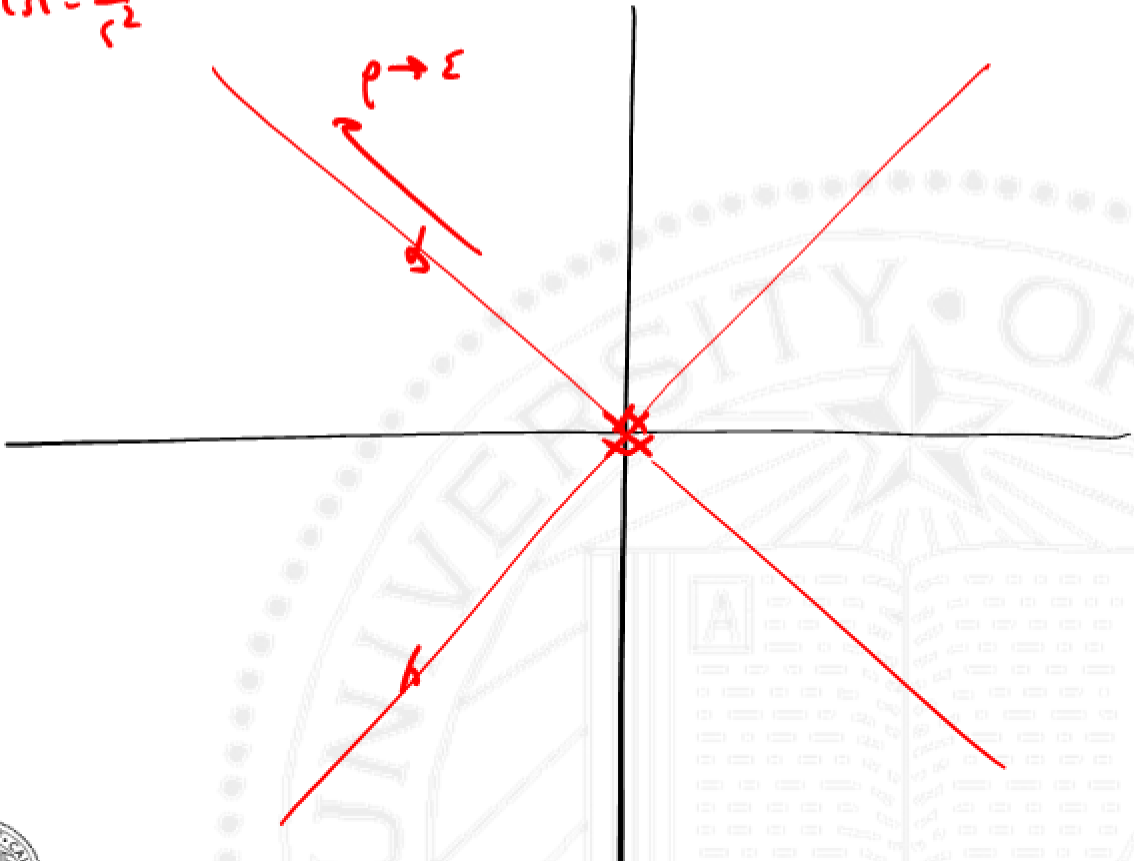


$$J = \int_0^\infty (\rho y^2 + u^2)\, dt$$

$$\frac{1}{s-a}$$

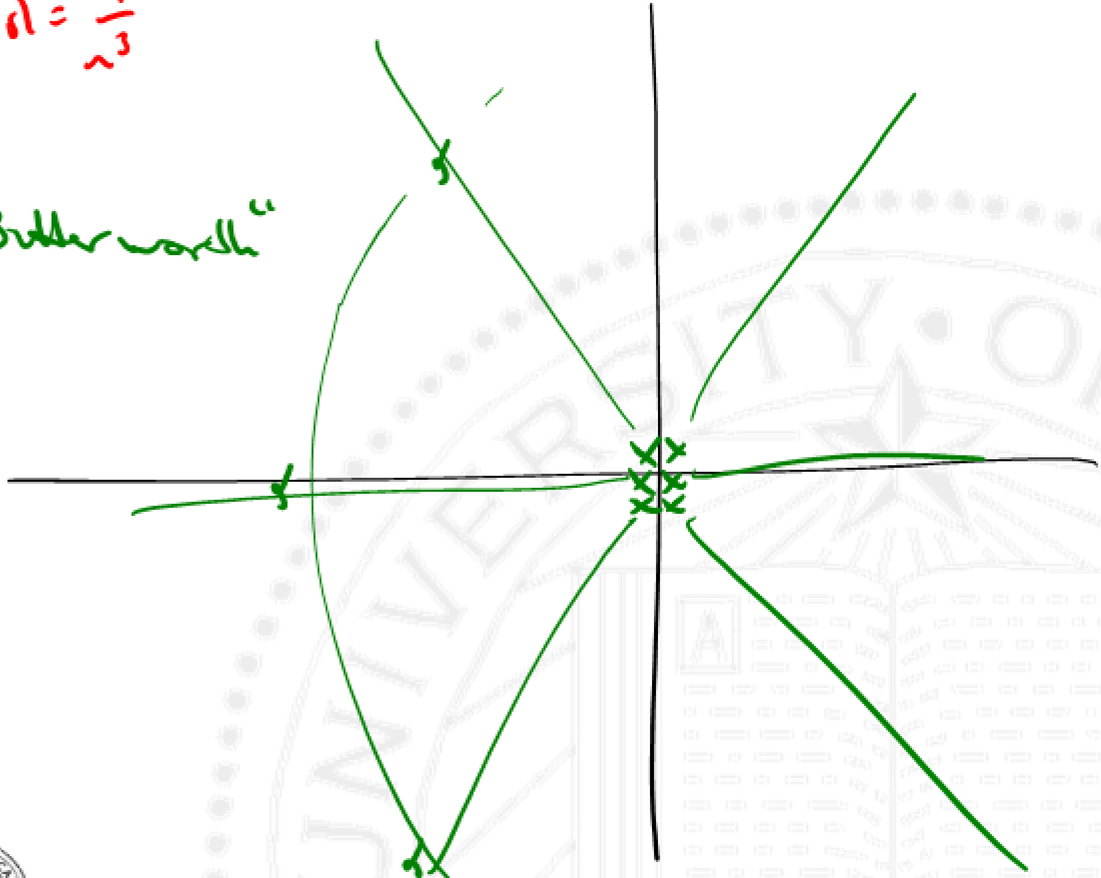


$$G(s) = \frac{1}{s^2}$$



$$G(s) = \frac{1}{s^3}$$

"Butterworth"



Non-minimum phase zero "in RHP"