CMPE-242
Applied Feedback Control

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Quaternions

$a + b\hat{i} + c\hat{k} + dj$

Sir Carling Hamilton

$$\begin{bmatrix} q_0 \\ \hline q_1 \\ \vdots \\ q_4 \end{bmatrix}$$

$|q| = 1$

q^* — complement (−)

$q \otimes s$ — multiplication

$$\dot{q} = -\frac{1}{2}\underbrace{\Omega}_{4\times 4} q \qquad \Omega = \left[\begin{array}{c|c} 0 & \omega^T \\ \hline \omega & (\omega \times) \end{array}\right]$$

$$\omega = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$q_{k+1} = \underline{q_k + \dot{q}\Delta t}$$

$$q_{k+1} = \underbrace{e^{-\frac{1}{2}\Omega \Delta t}} q_k$$

matrix exponential

$$\underline{q} \longrightarrow \begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix} \longrightarrow \begin{matrix}\text{DCM}\\ 3\times 3\end{matrix}$$

$$\frac{m}{b} \longrightarrow \frac{m}{i}$$

$$\boxed{\frac{m}{i} = \underline{q}^* \otimes \begin{bmatrix}0\\ \frac{m}{b}\end{bmatrix} \otimes \underline{q}}$$

Announcements

1. Grading midterms
2. Office hours — this afternoon
3. Link for homework (7/8)

T.F: $\left.\frac{Y}{U}\right|_{\cancel{x_0} \to 0} = C\underbrace{[sI - A]^{-1}}B + D$

$$M^{-1} \triangleq \frac{\text{ADJOINT}(M)}{\text{DET}(M)}$$

$$C\left[\frac{\text{ADJOINT}(sI-A)}{\text{DET}(sI-A)}\right]B + D$$

Δ_{cl}: $\det(sI-A) = 0$

Eigenvalues: $Ax_0 = \lambda x_0 \longrightarrow (\lambda I - A)x_0 = 0$.

trivial solution: $x_0 = 0$.

if $x_0 \neq 0$ — $\det(\lambda I - A) = \phi$.

λ's are the eigenvalues of A corresponding to the eigenvectors x_0.

$$\frac{C\left[\text{ADJOINT}(sI-A)\right]B}{\text{DET}(sI-A)} \qquad \therefore \ \text{DET}(sI-A) = \phi \leftarrow \Delta(s)$$

$$\boxed{\text{Poles of the system} \Leftrightarrow \text{eigenvalues of } A.}$$

eigenvalues of A $\longleftrightarrow$ Poles of <u>TF</u>

$$\dot{x} = Ax + Bu$$

choose $u = -Kx$

$$y = Cx + Du$$

$$\dot{x} = Ax + B(-K)x = (A - BK)x$$

$$y = Cx + D(-K)x = (C - DK)x$$

closed loop poles of my system are

$$\boxed{eig(A - BK)}$$

STATE SPACE CONTROL

$$\mathrm{eig}(A - BK) \leftrightarrow \Delta_{cl}(s)$$

Choose K to put $\Delta_{cl}(s)$ in some desirable location.

Controller Canonical Form

$$\frac{Y}{U} = \frac{1}{s^n + a_1 s^{n-1} + \dots + a_n} \qquad \text{choose for } \underline{x} = \begin{bmatrix} y^{(n-1)} \\ \vdots \\ \ddot{y} \\ \dot{y} \\ y \end{bmatrix}$$

$$\frac{y}{u} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3} \qquad \text{choose } \underline{x} = \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix}$$

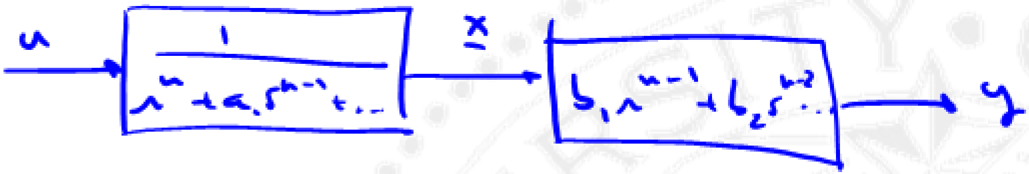
$$\underline{x} = \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix} \rightarrow \dot{\underline{x}} = \begin{bmatrix} \dddot{y} \\ \ddot{y} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \dot{y} \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ \hline 0 \\ 0 \end{bmatrix} u$$

$$\frac{y}{u} = \frac{1}{s^n + a_1 s^{n-1} + \dots a_n}$$

choose $\underline{x} = \begin{bmatrix} y^{(n-1)} \\ \vdots \\ \dot{y} \\ y \end{bmatrix}$

$$\dot{\underline{x}} = \left[\begin{array}{c|c} \multicolumn{2}{c}{-a_1 \quad -a_2 \quad \cdots \quad -a_n} \\ \hline I & 0_{\text{not square}} \end{array}\right] \underline{x} + \begin{bmatrix} 1 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$\frac{y}{u} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n}$$



$$\dot{\underline{x}} = \left[\begin{array}{ccc} -a_1 & \cdots & a_n \\ \hline 1 & & 0 \\ & \ddots & \vdots \\ & 1 & 0 \end{array}\right] \underline{x} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

Controller Canonical Form (CCF)

$$y = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix} \underline{x} + [0]\, u$$

$$\frac{Y}{U} = \frac{b_0 s^n + b_1 s^{n-1} + \dots}{s^n + a_1 s^{n-1} + \dots}$$

$$\begin{array}{r|l} & b_0 \\ \hline s^n + a_1 s^{n-1} + \dots & b_0 s^n + b_1 s^{n-1} + \dots \\ & -b_0 s^n - b_0 a_1 s^{n-1} - \dots - b_0 a_n \\ \hline & \underbrace{(b_1 - b_0 a_1)}_{c_1} s^{n-1} + \dots + \underbrace{(b_n - a_0 b_n)}_{c_n} \end{array}$$

$$\frac{Y}{U} = b_0 + \frac{c_1 s^{n-1} + \dots + c_n}{s^n + a_1 s^{n-1} + \dots}$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \underline{x} + (b_0) u$$

$$\underline{\dot{x}} = \left[\begin{array}{ccc} a_1 & \cdots & a_n \\ \hline 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array}\right] \underline{x} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$\dot{x}_1 \rightarrow \boxed{\frac{1}{s}} \rightarrow x_1 \qquad \dot{x}_2 \rightarrow \boxed{\frac{1}{s}} \rightarrow x_2 \qquad \dot{x}_3 \rightarrow \boxed{\frac{1}{s}} \rightarrow x_3$$

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -b & -c \\ d & -e \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

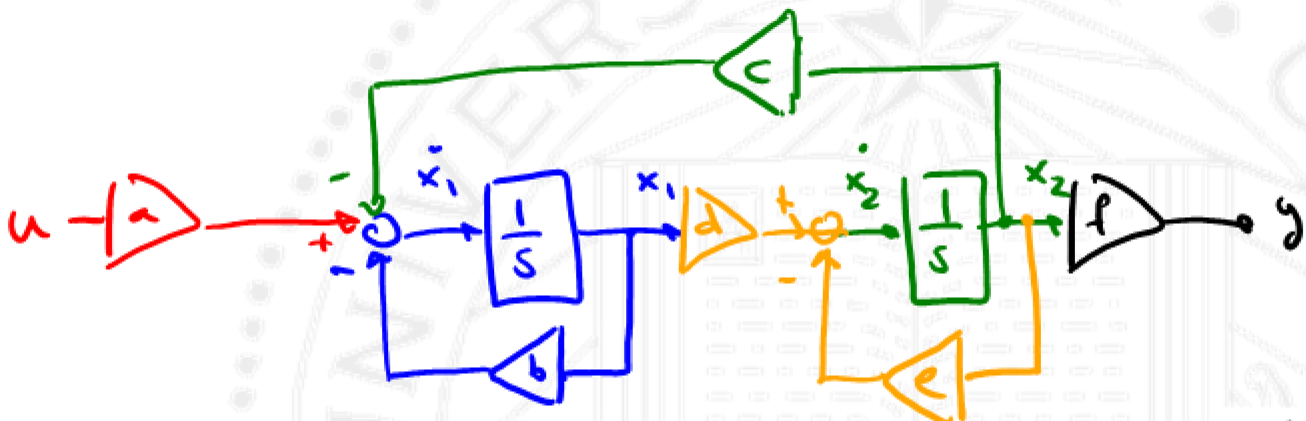
$$\begin{array}{cc} A & B \\ \begin{bmatrix} -b & -c \\ d & -e \end{bmatrix} & \begin{bmatrix} a \\ 0 \end{bmatrix} \end{array}$$

$$C \begin{bmatrix} 0 & f \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix} D$$

$$\dot{x}_1 = -bx_1 - cx_2 + au$$

$$\dot{x}_2 = dx_1 - ex_2 + 0u$$

$$y = fx_2 + 0u$$



STATE VARIABLES ARE NOT UNIQUE

$$\dot{x} = Ax + Bu \qquad x = Tz \qquad \dot{z} = T^{-1}ATz + T^{-1}Bu$$

$$y = Cx + Du \qquad z = T^{-1}x \qquad y = CTz + Du$$

$$\boxed{\det(sI - \tilde{A}) = 0} \qquad I \rightarrow T^{-1} I T$$

$$\det(sT^{-1}IT - T^{-1}AT) = \phi = \det(T^{-1})\det(sI - A)\det(T)$$

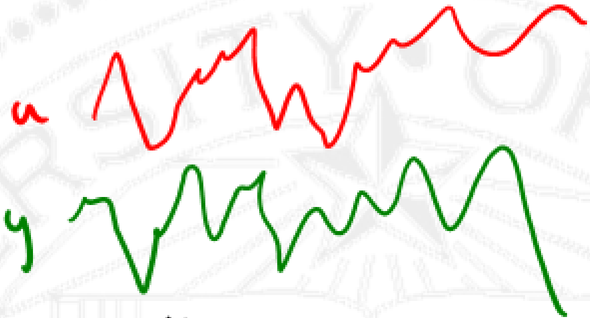
$$\det(T) \neq 0 \qquad \det(sI - A) = \phi.$$

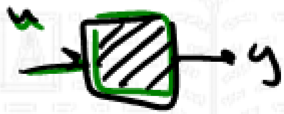
$$\det(T^{-1}) \neq 0.$$

eigenvalues of A $\overset{\text{SAME}}{\longleftrightarrow}$ eigenvalues of $T^{-1}AT$

$\mathrm{eig}(A)$ = poles of the system

$\mathrm{eig}(T^{-1}AT)$





$$\left[\begin{array}{c|c} T^{-1}\Phi T & T^{-1}\Gamma \\ \hline HT & D \end{array}\right] \longrightarrow \left[\begin{array}{ccc} \ddots & & \\ & \ddots & \\ & & \ddots \end{array}\right]$$

$$\left[\begin{array}{c|c} \Phi & \Gamma \\ \hline H & D \end{array}\right]$$

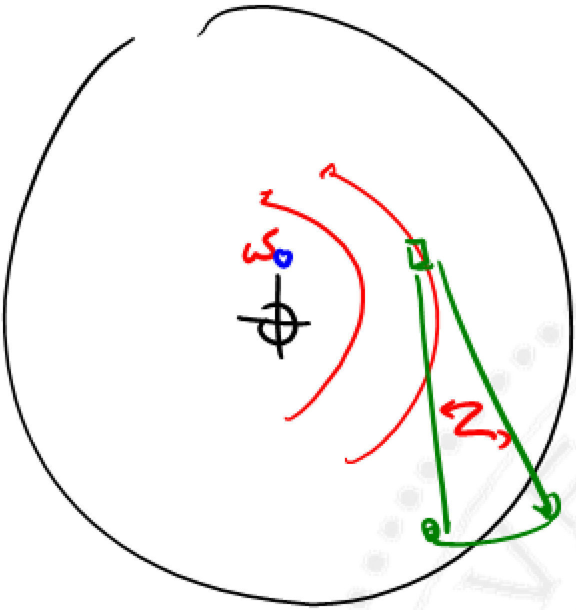
Parameter Identification

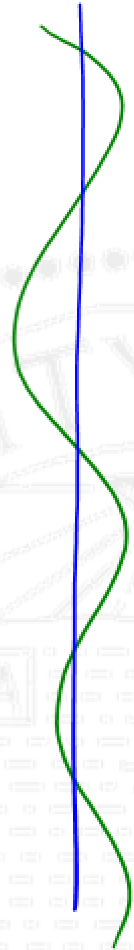
$$\dot{x} = \overset{\wedge}{\begin{bmatrix} \emptyset & \\ & \emptyset \\ & \emptyset \end{bmatrix}}^{*}$$

$$\begin{bmatrix} \dot{x} \\ \hline p \end{bmatrix} = \left[\begin{array}{c|c} A & \text{mx} \\ \hline \phi & \phi \end{array}\right] \begin{bmatrix} x \\ \hline p \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ p \end{bmatrix} = f\left(\begin{bmatrix} x \\ \hline p \end{bmatrix}, u \right) \longrightarrow A \simeq \left. \frac{\partial f}{\partial x} \right|_{\underline{x}}.$$

Extended Kalman Filter



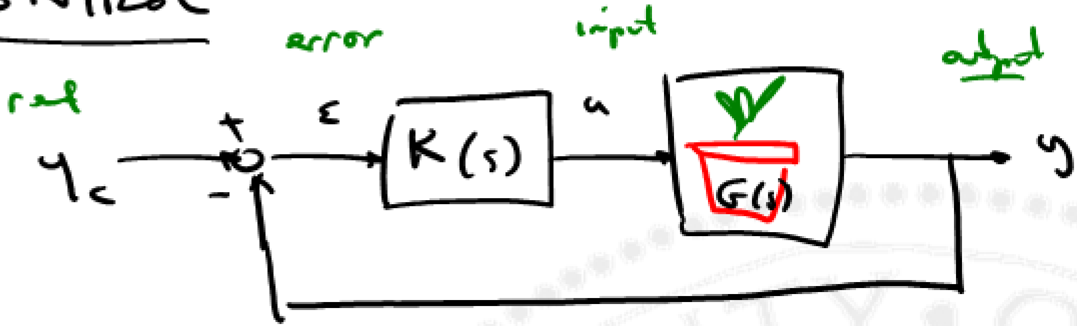


$A \sin(w_0 t + \phi)$

$$\begin{bmatrix} 0 & w_0 \\ -w_0 & 0 \end{bmatrix}$$

$-A \sin(w_0 t + \phi)$

CONTROL



$$U(s) = K(s)\left[Y(s) - R(s)\right]$$

$$G(s)\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

ref. state

$$K(s) = -k(\underline{x} - \underline{x}_0)$$

$$\dot{x} = Ax + Bu = Ax + B\left[-k(x - x_0)\right] = Ax - BKx + BKx_0$$

$$\underline{\dot{x}} = \underbrace{\left[A - BK\right]}_{\tilde{A}}\underline{x} + \underbrace{BK}_{\tilde{B}}\,\underbrace{\underline{x}_0}_{u}$$

$$y = Cx + Du = Cx + D\left[-Kx + Kx_0\right] = \underbrace{(C + DK)}_{\tilde{C}}\underline{x} + \underbrace{DK}_{\tilde{D}}\underline{x_0}$$

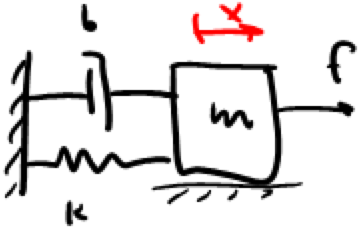
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$\longrightarrow u = -K(x - x_0) \longrightarrow$

$$\underline{\dot{x}} = \tilde{A}\underline{x} + \tilde{B}x_c$$
$$y = \tilde{C}\underline{x} + \tilde{D}x_c$$

$\mathrm{eig}(A)$ – poles of my plant

$\mathrm{eig}(\tilde{A}) = \mathrm{eig}(A - BK)$ – poles of closed loop system.



$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{f}{m}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u$$

$$\frac{d}{dt}\begin{bmatrix}x\\ \dot{x}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix} f$$

$$u = -kx$$

$$u = -\begin{bmatrix}k_1 & k_2\end{bmatrix}\begin{bmatrix}\dot{x}\\ x\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix}\dot{x}\\ x\end{bmatrix} = \begin{bmatrix}-2\zeta\omega_n & -\omega_n^2\\ 1 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ x\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix} u$$

$$A - BK = A - \begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix} = A - \begin{bmatrix}k_1 & k_2\\ 0 & 0\end{bmatrix}$$

$A - BK$

$$\begin{bmatrix} -2\zeta\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} -(2\zeta\omega_n+k_1) & -(\omega_n^2+k_2) \\ \hline 1 & 0 \end{array}\right]$$

$\Delta_{cl} = -1, -1$ $\quad \Delta_{cl} = s^2 + 2s + 1$

$$\det(sI - (A-BK)) = \det\left[\begin{array}{c|c} s + 2\zeta\omega_n + k_1 & \omega_n^2 + k_2 \\ \hline -1 & s \end{array}\right]$$

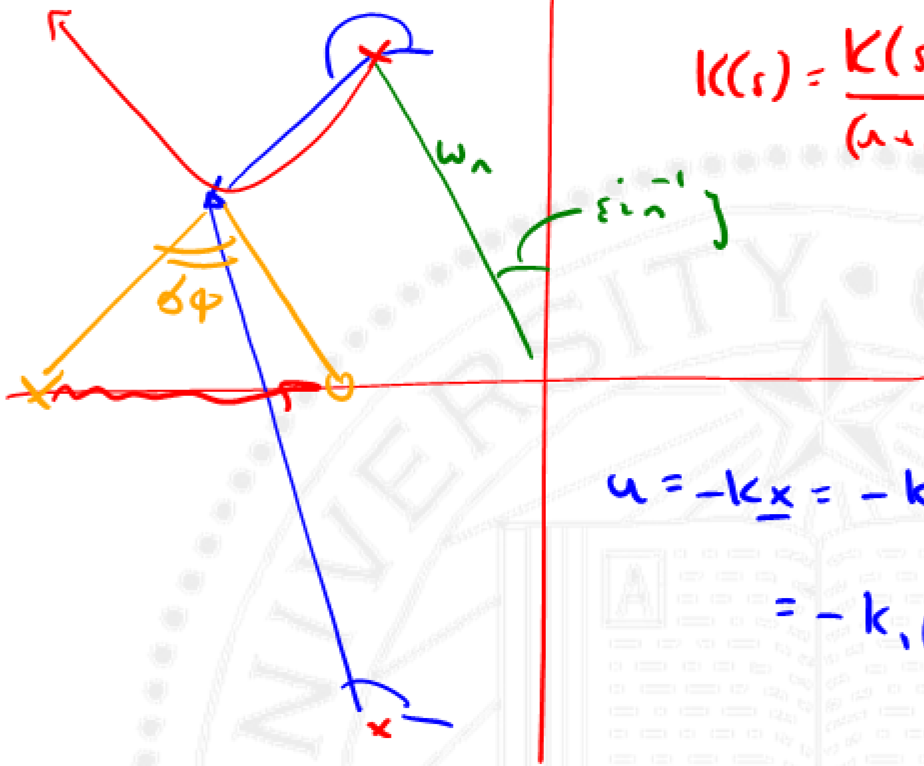
$$s^2 + \underbrace{(2\zeta\omega_n + k_1)}_{+2}s + \underbrace{(\omega_n^2 + k_2)}_{1} = \phi.$$

$$2\zeta\omega_n + k_1 = +2$$

$$\boxed{k_1 = 2 - 2\zeta\omega_n}$$

$$\omega_n^2 + k_2 = 1 \quad \therefore \quad \boxed{k_2 = 1 - \omega_n^2}$$

$\delta\phi = ?$

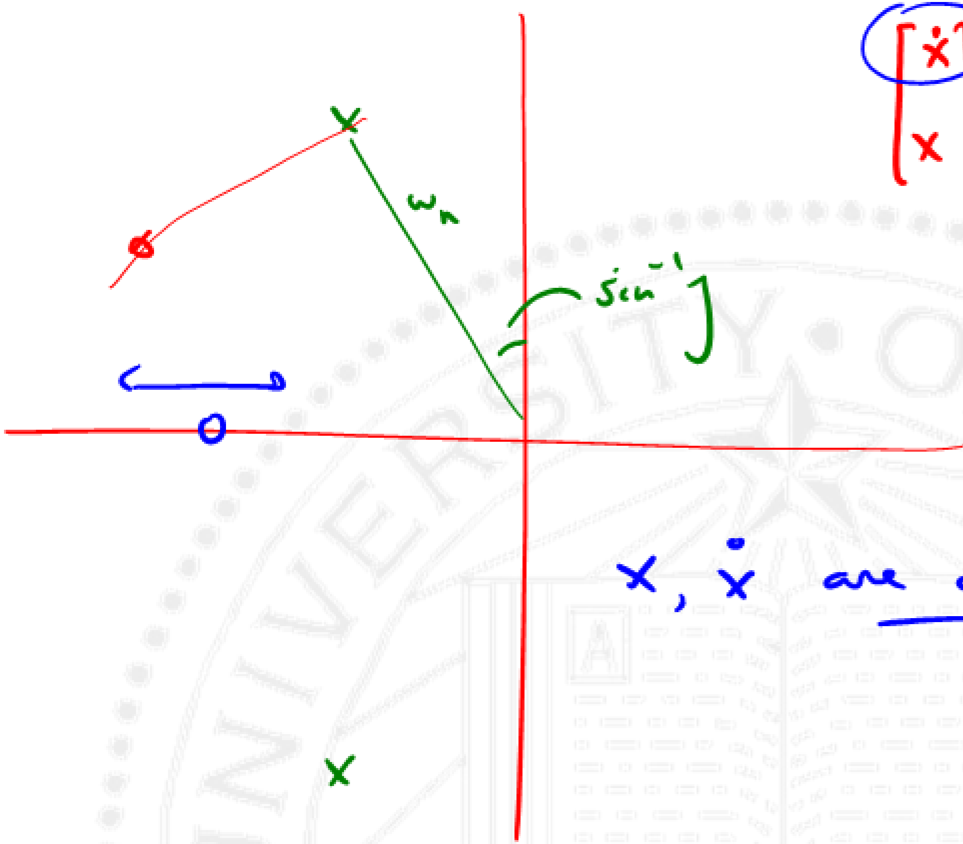


$$K(s) = \frac{K(s+z)}{(s+p)} = \frac{u}{e}$$

$$u = -K\underline{x} = -k_1 \dot{x} - k_2 x$$

$$= -k_1\left(s + \frac{k_2}{k_1}\right)x$$

$$\begin{bmatrix}\dot{x}\\ x\end{bmatrix}$$



$x, \dot{x}$ are different

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$K = [k_1 \quad k_2 \quad k_3]$$

$$A - BK = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [k_1 \quad k_2 \quad k_3]$$

$$A - BK = \begin{bmatrix} -(a_1+k_1) & -(a_2+k_2) & -(a_3+k_3) \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

controller canonical form

choose K — "piece of cake"

$$\Delta_{cl} = s^3 + (a_1+k_1)s^2 + (a_2+k_2)s + (a_3+k_3) = 0$$

$$A - BK = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} a_1 - b_1 k_1 & a_2 - b_1 k_2 \\ a_3 - b_2 k_1 & a_4 - b_2 k_2 \end{bmatrix}$$

$$\det(\lambda I - (A - BK)) = 0$$

$$\lambda^2 + \left[(a_1 - b_1 k_1) + (a_2 - b_2 k_2)\right]\lambda + \left[(a_1 - b_1 l_1)(a_3 - b_2 k_2) + (a_3 - b_1 k_2)(a_4 - b_1 k_2)\right] = \phi.$$

2 eqn 2 unknowns. $\longrightarrow k_1, k_2$

$x = Tz$

$\uparrow$ controller canonical form

$$T^{-1}AT = \begin{bmatrix} & -a & \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$u = -Kz$

$u = \underbrace{-KT^{-1}}_{K^*}x$

Appendix E

Ackerman's Formula

$K = \text{acker}(A, B, \underset{\text{desired poles}}{p})$

$K = \text{place}(A, B, p)$

$x = Tz$

orig. ... c.c.f.

$u = -kT^{-1}x$

c.c.f.

$$K = [0 \;—\; 0 \;\; 1]\,\mathcal{C}^{-1}\alpha_0(A)$$

$$\mathcal{C} \triangleq [B \;\; AB \;\; A^2B \;\cdots\; A^{n-1}B]$$

controllability matrix

if rank $(\mathcal{C}) = n$

put poles anywhere I want

$$\alpha_0(A) \triangleq A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots \alpha_n I$$

$$\Delta_{cl}(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots$$

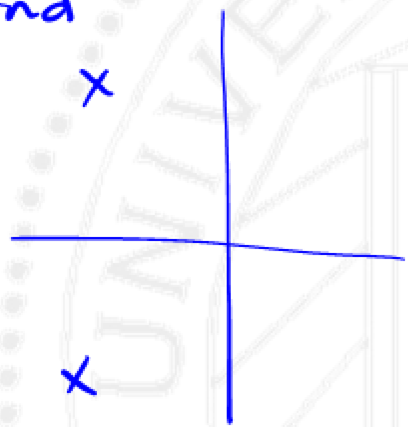
$$\mathcal{C} \triangleq \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

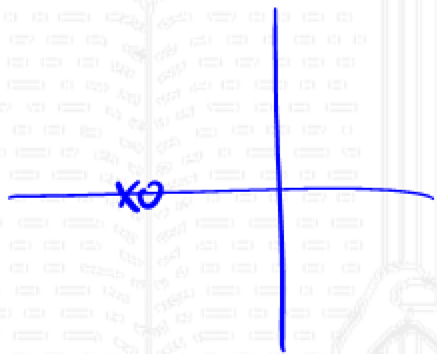
ctrb(A, B)

$\text{rank}(\mathcal{C}) = n$ for control.

Condition number of $\mathcal{C} \triangleq \dfrac{\sigma_{max}(\mathcal{C})}{\sigma_{min}(\mathcal{C})}$

rcond



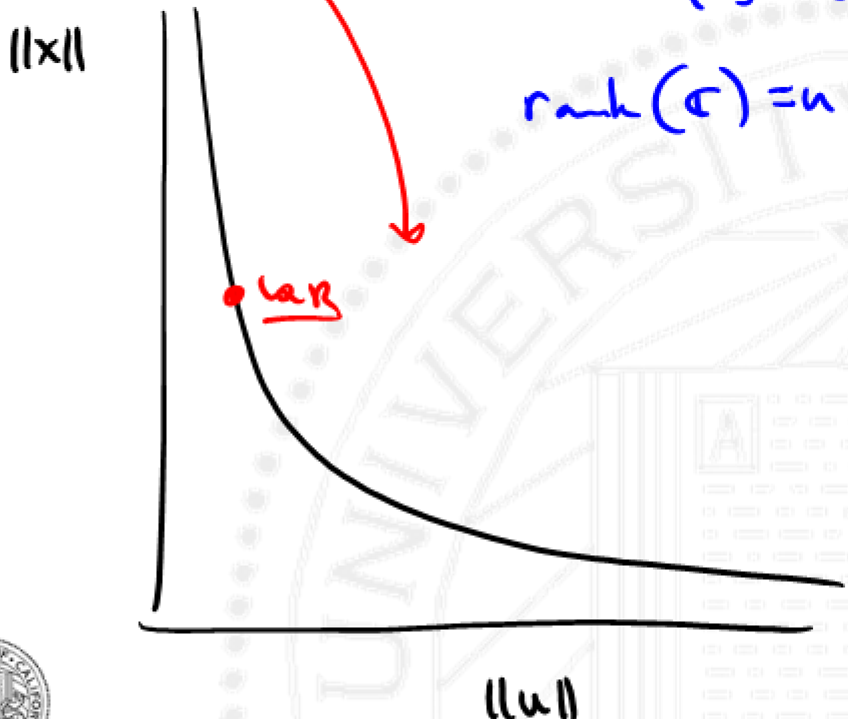


Pole Placement

$\dot{x} = Ax + Bu \qquad u = -Kx$

$\mathcal{C} = \left[B \;\; AB \;\; A^2B \;\cdots\; A^{n-1}B \right]$

$\mathrm{rank}(\mathcal{C}) = n$ put poles down w/o restriction



State Space Control

Control — let $u = -K\underline{x}$

choose K to put $\text{eig}(A-BK)$ where I want

$$\dot{x} = Ax + Bu \qquad \underline{u = -Kx}$$

$$\text{eig}(A-BK) - \Delta_d(s)$$

$$\mathcal{C} = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B] \left\langle \begin{array}{l} \text{rank}(\mathcal{C}) = n \\ \text{rcond}(\mathcal{C}) \approx 1. \end{array} \right.$$

$\boxed{\text{FULL STATE FEEDBACK}}$