CMPE-242
Applied Feedback Control

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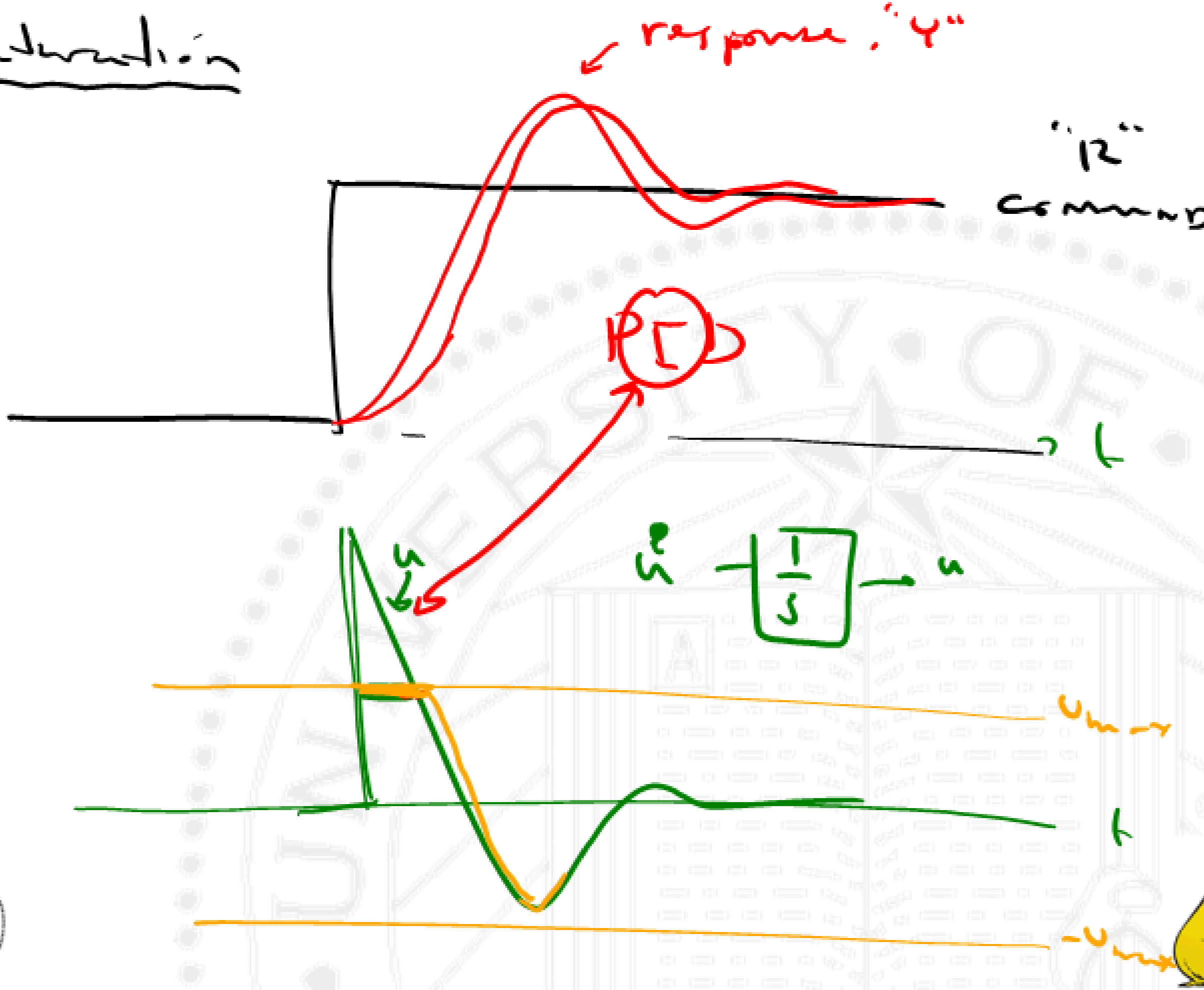
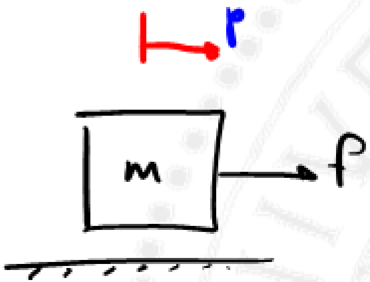
Saturation



What is the state

$\underline{x} \in \mathbb{R}^n$ n x 1 vector

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$



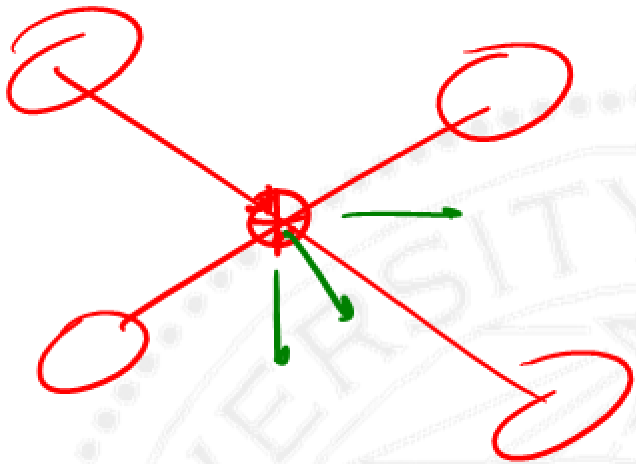
$m\ddot{p} = f$

$$\dot{\underline{x}} = \underline{A}x + \underline{B}\underbrace{u}_{f}$$

$$\dot{\left[\begin{array}{c} p \\ \hline \dot{p} \end{array}\right]} =$$

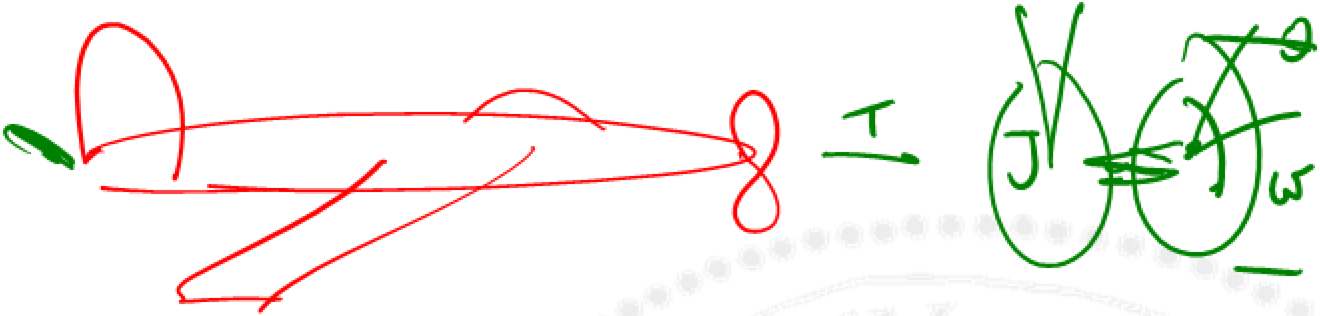
$$\ddot{x} = \frac{1}{m} f_{ext}$$ body

inertial

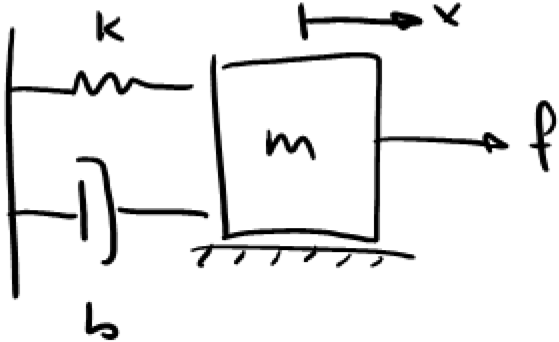


$$\ddot{\theta} = \frac{1}{I} M_{ext}$$

$$\begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \\ \psi \\ \theta \\ \phi \\ \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$



$$\longrightarrow \begin{bmatrix} x \\ u \\ \theta \\ \dot{\theta} \end{bmatrix} \qquad \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}$$



$$m\ddot{x} + b\dot{x} + kx = f$$

$$\ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x + \frac{f}{m}$$

$$\dot{x} = v$$

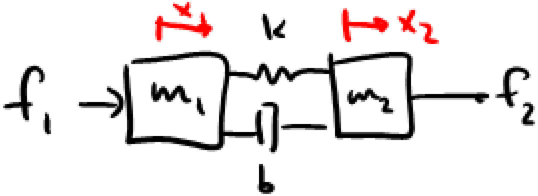
$$\dot{v} = \ddot{x}$$

$$\begin{bmatrix} x \\ v \end{bmatrix}$$

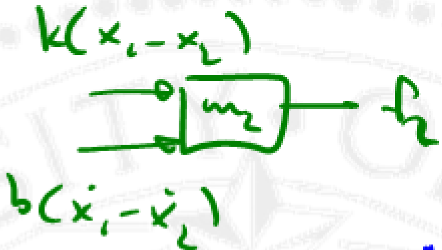
$\uparrow$
state

$$\dot{\underline{x}} = \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ \ddot{x} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}^{A}\begin{bmatrix} x \\ v \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}^{B} F$$

$$\underline{y} = \begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C}\underbrace{\begin{bmatrix} x \\ v \end{bmatrix}}_{\uparrow\ \text{physics}} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{D} F$$



$$\frac{x_1}{F_1} \quad \frac{x_1}{F_2} \quad \frac{x_2}{F_1} \quad \frac{x_2}{F_2}$$



$$m_1 \ddot{x}_1 = F_1 + k(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1)$$

$$m_2 \ddot{x}_2 = F_2 + k(x_1 - x_2) + b(\dot{x}_1 - \dot{x}_2)$$

$$\underline{x} = \begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix}$$

$$\underline{\dot{x}} = \begin{bmatrix} \ddot{x}_1 \\ \dot{x}_1 \\ \ddot{x}_2 \\ \dot{x}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{b}{m_1} & -\frac{k}{m_1} & \frac{b}{m_1} & \frac{k}{m_1} \\ 1 & 0 & 0 & 0 \\ \frac{b}{m_2} & \frac{k}{m_2} & -\frac{b}{m_2} & -\frac{k}{m_2} \\ 0 & 0 & 1 & 0 \end{bmatrix}}^{A} \begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \\ 0 & 0 \end{bmatrix}}^{B} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{D} \begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}^{C} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Similarity Transform

$$z = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 - x_1 \\ \dot{x}_2 - \dot{x}_1 \end{bmatrix}$$

T (and T^{-1}) exists such that

$$\underbrace{\underline{x}}_{n\times 1} = \overbrace{T}^{n\times n} \underbrace{z}_{n\times 1} \qquad z = T^{-1}x \qquad T\dot{z} = T\dot{z}$$

$$\dot{z} = T^{-1}\dot{x}$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$T^{-1}\left[T\dot{z} = ATz + Bu\right]$$

$$y = CTz + Du$$

$$\dot{x} = Ax + Bu \qquad x = Tz$$

$$y = Cx + Du \qquad \dot{x} = T\dot{z}$$

$$\downarrow$$

$$\dot{z} = \overbrace{T^{-1}AT}^{\tilde{A}}\, z + \overbrace{T^{-1}B}^{\tilde{B}}\, u$$

$$y = \underbrace{CT}_{\tilde{C}}\, z + \underbrace{D}_{\tilde{D}}\, u$$

$$\overset{\underline{x}}{\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]} \longleftrightarrow \overset{\underline{z}}{\left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right]}$$

Center of mass

$$\frac{m_1}{m_2+m_1}x_1 + \frac{m_2}{m_2+m_1}x_2 \triangleq x_{cm}$$

$$\dot{x}_{cm}$$

$$y = \left[\begin{array}{c|c|c|c} 0 & \frac{m_1}{m_1+m_2} & 0 & \frac{m_2}{m_1+m_2} \\ \hline \frac{m_1}{m_1+m_2} & 0 & \frac{m_2}{m_1+m_2} & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_1 - x_2 \\ \dot{x}_1 - \dot{x}_2 \end{bmatrix}}_{\text{"}z\text{"}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}}_{T^{-1}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix}}_{\text{"}x\text{"}}$$

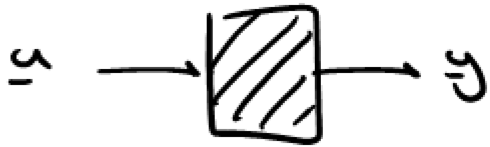
$$T^{-1} T = I$$

$$\begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}}_{T} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_1 - x_2 \\ \dot{x}_1 - \dot{x}_2 \end{bmatrix}$$

✓

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \boxed{1 \quad 0 \quad -1 \quad 0} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

System Identification



"Black Box"

$$x_{k+1} = \Phi x_k + \Gamma u_k$$

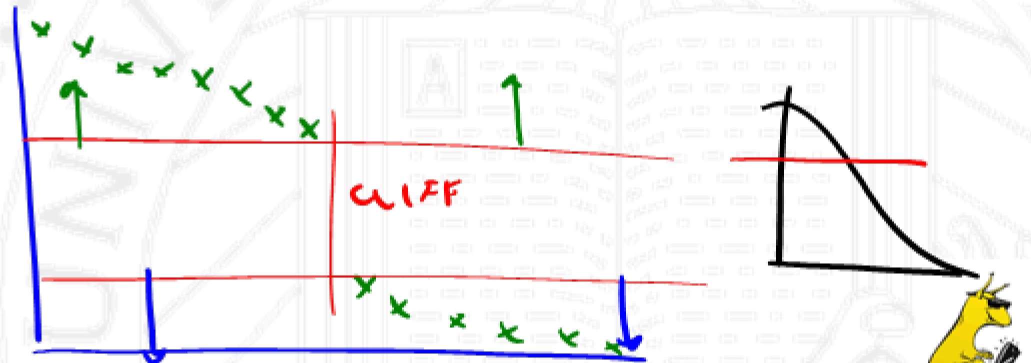
$$y_k = H x_k + D u_k$$

$$\left[\begin{array}{c|c} \Phi & \Gamma \\ \hline H & D \end{array}\right]$$

300ths order

BALREAL

SVD$(\Phi^T\Phi)$



$$\left.\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ y &= C\underline{x} + D\underline{u} \end{aligned}\right\} \text{ TIME DOMAIN EQUATIONS}$$

$$\mathcal{L}\left\{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \longrightarrow \begin{aligned} sX(s) - \cancelto{0}{x_0} &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

$$sX(s) - AX(s) = BU(s) + \cancelto{0}{x_0}$$

$$[sI - A]X(s) = BU(s) + \cancelto{0}{x_0}$$

$$X(s) = [sI - A]^{-1} BU(s) + [sI - A]^{-1}\cancelto{0}{x_0}$$

$$Y(s) = C[sI - A]^{-1} BU(s) + C[sI - A]^{-1}\cancelto{0}{x_0} + DU(s)$$

$$\boxed{\frac{Y}{U}(s) = C[sI - A]^{-1}B + D}$$

TRANSFER FN

SIMILARITY TRANSFORM

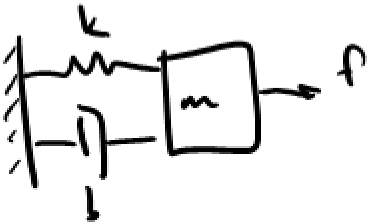
$$\begin{cases} \tilde{A} = T^{-1}AT \\ \tilde{B} = T^{-1}B \\ \tilde{C} = CT \\ \tilde{D} = D \end{cases}$$

$$\frac{Y}{U} = CT[\underbrace{sI}_{T^{-1}T} - T^{-1}AT]^{-1}T^{-1}B + D$$

$$\frac{Y}{U} = CT(\overset{A}{T^{-1}}\overset{B}{(sI - A)}\overset{C}{T})^{-1}T^{-1}B + D$$

$$\underline{(ABC)^{-1} = C^{-1}B^{-1}A^{-1}}$$

$$\frac{Y}{U} = C\underline{TT^{-1}}[sI - A]^{-1}\underline{TT^{-1}}B + D = \underline{C[sI - A]^{-1}B + D}$$



$$A = \begin{bmatrix} -\frac{b}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} v \\ x \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad D = 0$$

$$\frac{Y}{U} = C(sI - A)^{-1}B + D$$

$$\underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C}\left(\underbrace{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}}_{sI} - \underbrace{\begin{bmatrix} -\frac{b}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix}}_{A}\right)^{-1} \underbrace{\begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix}}_{B} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} (s + \frac{b}{m}) & k/m \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} + \phi.$$

$$\underbrace{\frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -k/m \\ 1 & s+\frac{b}{m} \end{bmatrix}}{s\left(s+\frac{b}{m}\right)+\frac{k}{m}}}_{[sI-A]^{-1}} \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} + 0$$

$$\frac{Y}{U} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s}{m} \\ \frac{1}{m} \end{bmatrix}}{\left(s^2+\frac{b}{m}s+\frac{k}{m}\right)}$$

$$\boxed{\frac{\frac{1}{m}}{\left(s^2+\frac{b}{m}s+\frac{k}{m}\right)}}$$