HOMEWORK #9
DUE 16-MAR-2017

1. Put the following transfer functions into state space using the controller canonical form (use MATLAB to check your answers after you have done them by hand):
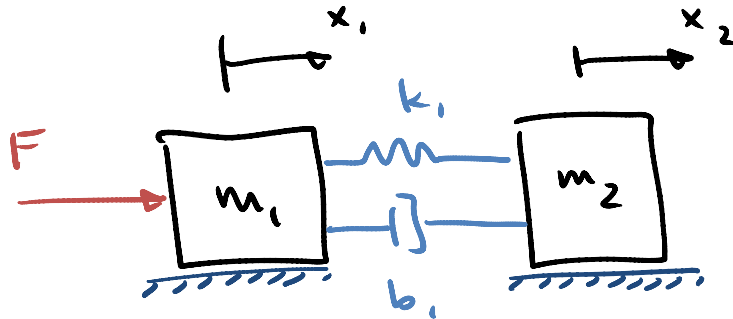
   a. $G(s) = \frac{1}{4s+1}$

   b. $G(s) = \frac{5(s/2+1)}{s/10+1}$

2. Given the following system, with zero initial conditions, find the steady state value of *x* to a step input:

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbf{u}$$

3. Consider the following two mass problem that we have been looking at in class. Let the mass of the first mass equal the second mass: $m_1 = 1\text{Kg}$, $m_2 = 1\text{Kg}$, $k_1 = 50$ N/m and $b_1 = 1$ Ns/m:

   a. Derive the state space equations using the state vector: $\mathbf{x} = [x_1 \quad \dot{x}_1 \quad x_2 \quad \dot{x}_2]^T$

   b. Find a similarity transform that will convert your answer in part (a) into a new set of equations with the state vector: $\mathbf{x}_{new} = [x_1 \quad \dot{x}_1 \quad x_{REL} \quad \dot{x}_{REL}]^T$, where x_{rel} refers to the distance between the two masses (x_2-x_1).

      i. What is the similarity transform: T and/or T^{-1}
      ii. What are the new state matrices (A,B,C,D)?

For parts (c) – (g) use the state: $\mathbb{x} = [x_1 \quad \dot{x}_1 \quad x_2 \quad \dot{x}_2]^T$

    c. Find the feedback gain matrix, $u = -K\mathbb{x}$, that will place the closed loop poles at a place that gives nice low frequency response and damps the high frequency mode. Use the pole locations: s_{des} = -3±2.5j, and -5±10j.

    d. Compute the initial condition response of the closed loop system of Part (c) with an initial state vector $\mathbb{x}_o = [1\,0\,0\,0]^T$. Plot the time histories $x_2(t)$ and u(t).

    e. Use the Symmetric Root Locus (SRL) to find a new set of desired pole locations that will have roughly the same low-frequency closed loop pole locations as in Part (c).

    f. Find a new feedback gain matrix, $u = -K\mathbb{x}$, that will put the closed loop poles at the locations found in part (e).

    g. Plot the initial condition responses of the new system as you did in Part (d), and compare both of them (over plot the two responses). Note the difference in the amount of control effort required.

Addition of an Estimator: Everything done in the parts above assumed we could measure all of the states. That is, you used the control, $u = -K\mathbb{x}$. Let's now assume that you can only have one measured quantity, y = x₂, and design an estimator that will allow the use of the control $u = -K\hat{\mathbb{x}}$.

    h. Design an estimator gain matrix, **L**, that will place the estimator roots at 10*[-10 -11 -12 -13]^T. These roots are roughly *10×* faster than the closed loop controller poles (I chose these rather than four poles at -10 to avoid the repeated roots).

    i. Find the state matrices that describe the combined controller and estimator for the regulator problem (no reference input). That is, the combined state is $\mathbb{x}_C = [\mathbb{x} \quad \hat{\mathbb{x}}]$, and the control is: $u = -K\hat{\mathbb{x}}$. Verify that the closed loop roots of this system (eigenvalues of the combined A matrix) are the closed loop controller poles and the closed loop estimator poles.

    j. Compute the initial condition response of the closed loop system from Part (i), and use the initial state: $\mathbb{x}_o = [1\,0\,0\,0 \mid 0\,0\,0\,0]^T$. Again, plot the time histories $x_2(t)$ and u(t).

    k. Repeat Parts (h) – Parts (j), but this time use estimator roots at 0.1*[-10 -11 -12 -13]^T. The point of this is to show you that the estimator roots can dominate the response if you let them get into the same frequency range at the controller response.

    l. Find the equivalent compensator, *u(s) = K(s)y(s)* (where y = x₂) of the system using the **K** from part (f) and the **L** from part (h). This K(s) should look familiar, it will have a zero to deal with the two poles at the origin, and a notch to deal with the high frequency lightly damped poles. It will also have some poles and zeros off at "infinity."

Advanced State Space methods: these parts will be very similar to what is going to be on the exam, so you might as well try them all.

    m. Find the feedback gain matrix, $u = -K\mathbb{x}$, that will minimize the cost function $J = \int_0^\infty y^T Q y + \rho u^2 \, dt$. Adjust ρ until you get about the same low frequency closed loop pole locations as in Part (e). Note that these "optimal" gains should be very close to

what you found in Part (f). Use the MATLAB function "LQR." You could also try using Bryson's rule to design your gains.

n. Add in reference inputs. Design the N_x and N_u matrices that will cause the system you designed in Part (f) to track a reference step command in $y=x_2$. That is, C = $[0\ 0\ 1\ 0]^T$. Find the matrices of the closed loop system. That is, the input r = y_c = x_{2c}.

Design in the discrete domain. We are going to redo the problem, but this time sampled in the digital domain:

o. Find the Φ, Γ, H, and D matrices that correspond to the system in Part (a) with a sample rate of 20Hz (T = 0.05).

p. Find the closed loop pole locations in the z-domain corresponding to the desired pole locations in Part (e).

q. Find the feedback gain matrix, $u = -K\mathbb{x}$, for this discrete domain system.

r. Compute the initial condition response of this discrete domain closed loop system. Again, use $\mathbb{x}_o$ = $[1\ 0\ 0\ 0]^T$, and plot the time histories of x_2 and u. Compare this with the continuous time systems responses in Part (g).

s. Repeat Parts (o) – Parts (r) with slower sample rates. How big can T get before the responses start to diverge?