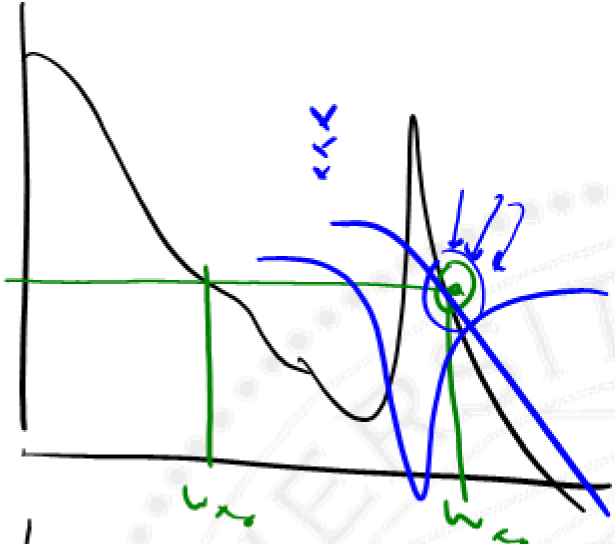
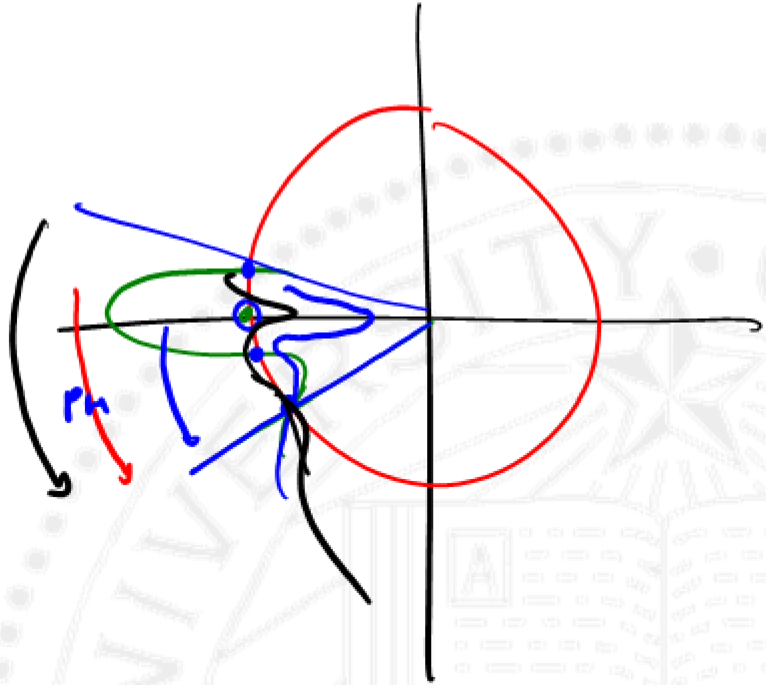
CMPE-242
Applied Feedback Control

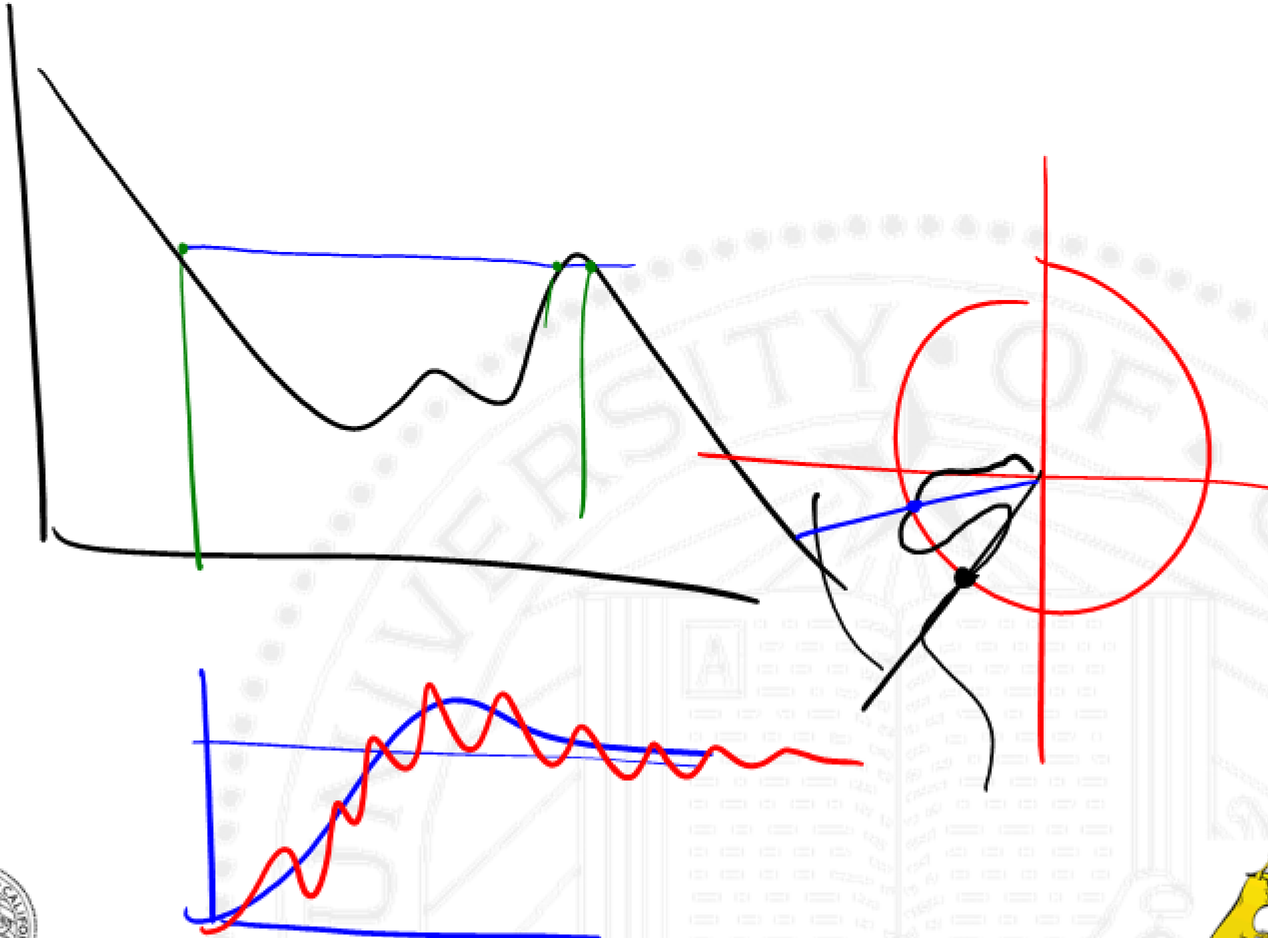
Gabriel Hugh Elkaim

Winter 2016



ω_m
ω_c





STATE VARIABLES ARE $\boxed{\text{NOT}}$ UNIQUE

$$\dot{x} = Ax + Bu \qquad x = Tz \qquad z = T^{-1}x$$

$$y = Cx + Du \qquad T^{-1}\left[T\dot{z} = ATz + Bu\right]$$

$$\dot{z} = \underline{T^{-1}AT}\, z + \underline{T^{-1}B}\, u$$

$$y = \underline{CT}\, z + \underline{D}\, u$$

$$\boxed{\det(sI - A) = 0} \qquad I \to T^{-1}IT$$

$$\det(sI - T^{-1}AT) = 0 = \det\left[T^{-1}sIT - T^{-1}AT\right] =$$

$$\det\left[T^{-1}(sI - A)T\right] = \cancel{\det(T^{-1})}\,\det(sI - A)\,\cancel{\det(T)} = 0$$

eigenvalues of A $\xleftrightarrow{\text{same}}$ eigenvalues of $T^{-1}AT$.

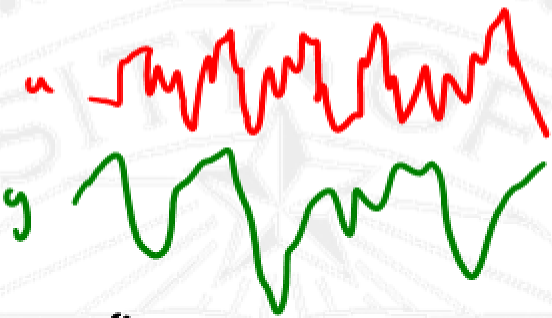
$$\operatorname{eig}(A) \longrightarrow \text{poles of system}$$

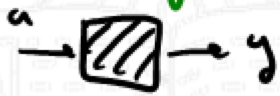
$$\|$$

$$\operatorname{eig}(T^{-1}AT)$$

$$\downarrow$$

$$\left[\begin{array}{c|c} T^{-1}\Phi T & T^{-1}\Gamma \\ \hline \underbrace{HT}_{\downarrow\ [\,\diagdown\,]\ s} & D \end{array}\right]$$





$$\left[\begin{array}{c|c} \Phi & \Gamma \\ \hline H & D \end{array}\right]$$

Parameter Identification

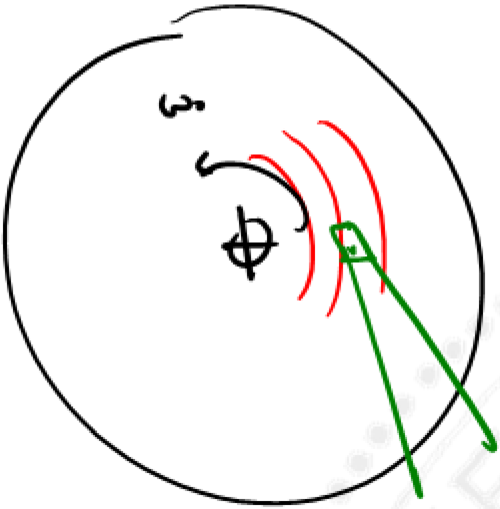
$$\begin{bmatrix} x \\ \hdashline p \end{bmatrix}^{\bullet} = \begin{bmatrix} A & \sim \\ 0 & \text{—} \; 0 \end{bmatrix} x + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} & p_1 & r_1 \\ & & r_2 \end{bmatrix} \begin{bmatrix} x \\ \hdashline p \end{bmatrix}$$

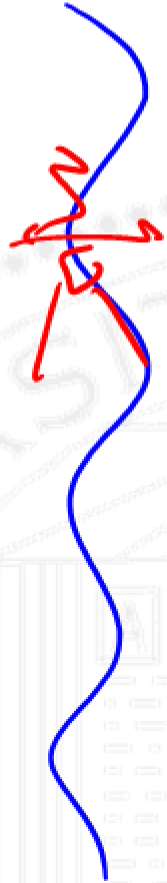
$$\begin{bmatrix} x \\ \hdashline r \end{bmatrix}^{\bullet} = f\left(\begin{bmatrix} x \\ \hdashline r \end{bmatrix}, u\right)$$

$$A \approx \left.\frac{\partial f}{\partial x}\right|_{x_0}$$

Extended Kalman Filter (EKF)



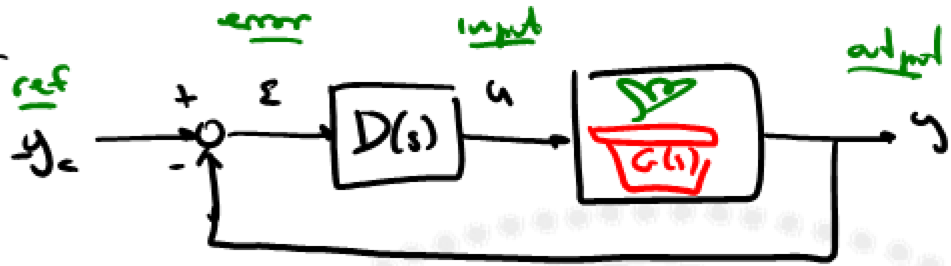
$A \sin(\omega t + \phi)$



$$\begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}$$

$-\Delta \sin(\omega t + \phi)$

CONTROL



$$U(s) = D(s)\left[Y(s) - R(s)\right]$$

$$G(s): \quad \dot{x} = Ax + Bu$$
$$\qquad\quad y = Cx + Du$$

ref state

$$D(s): \quad u = -K(x - x_c)$$

$$\dot{x} = Ax + Bu = Ax + B(-Kx + Kx_c) = Ax - BKx + BKx_c$$

$$\dot{\underline{x}} = \underbrace{(A - BK)}_{\tilde{A}}\underline{x} + \underbrace{BK}_{\tilde{B}}\underline{x}_c$$

$$y = Cx + D(-Kx + Kx_c) = \overbrace{(C - DK)}^{\tilde{C}}\underline{x} + \overbrace{DK}^{\tilde{D}}\underline{x}_c$$

$$\dot{x} = Ax + Bu \qquad \longrightarrow \qquad \dot{x} = \tilde{A}x + \hat{B}\underline{x}_c$$

$$y = Cx + Du \qquad \qquad \qquad y = \tilde{C}x + \tilde{D}x_c$$

eig(A) – poles of $G(s)$

eig$(\tilde{A})$ = eig$(A - BK)$

poles of CL. system.

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{f}{m}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u$$

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f$$

$$u = -K\underline{x}$$

$$u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix}\begin{bmatrix} \dot{x} \\ x \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} -2\zeta\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$A - BK = A - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix} = \left[A - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix}\right]\underline{x} + BK\underline{x}_c$$

$\underline{A - BK}$

$$\begin{bmatrix} -2\zeta\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} K_1 & k_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -(2\zeta\omega_n + k_1) & -(\omega_n^2 + k_2) \\ 1 & 0 \end{bmatrix}$$

$\Delta_{cl} = -1, -1$ $\qquad$ $\Delta_{cl} = s^2 + 2s + 1$

$$\det\left(sI - (A - BK)\right) = \begin{bmatrix} s + (2\zeta\omega_n + k_1) & \omega_n^2 + k_2 \\ -1 & s \end{bmatrix}$$

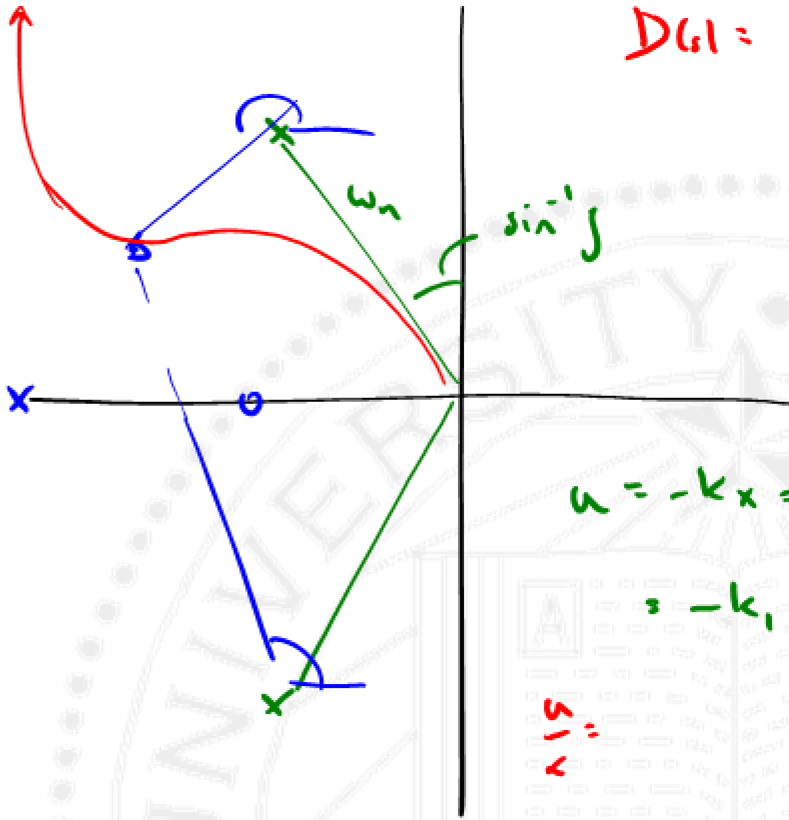
$$s^2 + (2\zeta\omega_n + k_1)s + (\omega_n^2 + k_2) = \phi$$

$$s^2 + 2s + 1 = \phi$$ σ damped

$$2\zeta\omega_n + k_1 = 2$$

$$\omega_n^2 + k_2 = 1$$

$$\boxed{\begin{aligned} k_1 &= 2 - 2\zeta\omega_n \\ k_2 &= 1 - \omega_n^2 \end{aligned}}$$

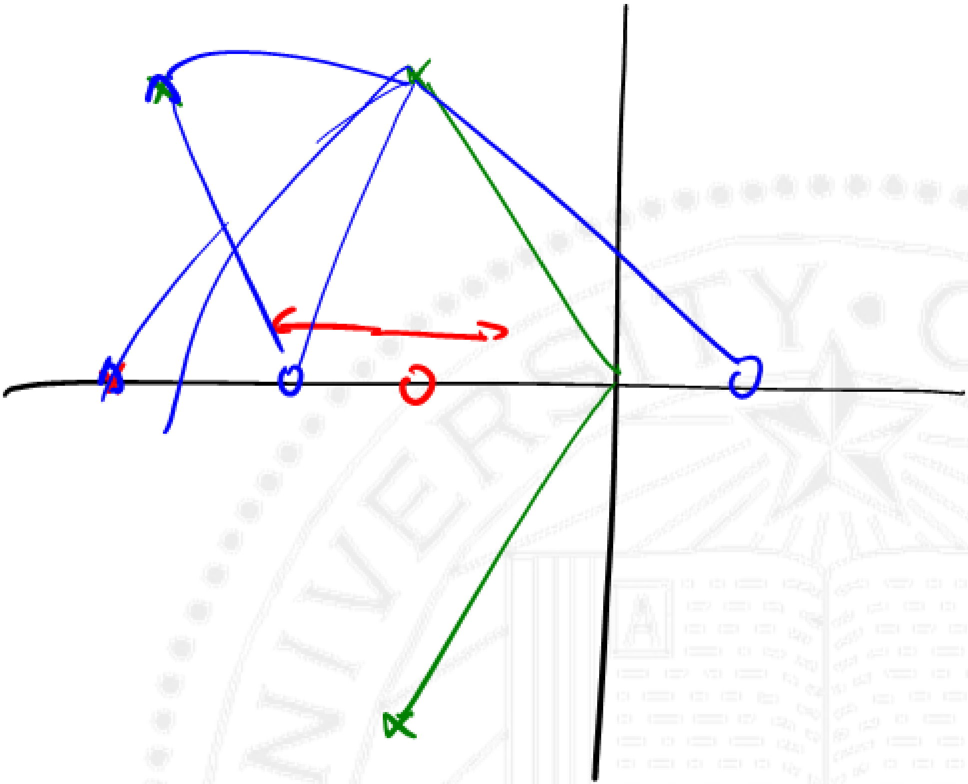


$$D(s) = \frac{K(s+z)}{s+p} = \frac{u}{e}$$

$$u = -Kx = -k_1\dot{x} - k_2 x$$

$$= -k_1\left(s + \frac{k_2}{k_1}\right)x$$

$$\frac{u}{e} =$$



$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$\leftarrow$ directly means the derivative.

$x, \dot{x}$ are different

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$K = (k_1 \; k_2 \; k_3)$$

$$A - BK = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} -(a_1+k_1) & -(a_2+k_2) & -(a_3+k_3) \\ 1 & & \\ & \ddots & 1 \end{bmatrix}$$

Controller canonical form

choose K → "pick a care"

$$\Delta_{cl} = s^3 + (a_1+k_1)s^2 + (a_2+k_2)s + (a_3+k_3)$$

$x = T z$

controller canonical form

$$T^{-1} A T = \begin{bmatrix} a_1 & \cdots & a_n \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$u = -K z$.

Appendix F.

$u = -\underbrace{K T^{-1}}_{K'} \underline{x}$

Ackermann's Formula $\leftarrow$ K = acker (A, B, P)

desired poles.

No repeated poles

"place"

K = place (A, B, P)

$$A - BK = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} a_1 - b_1 k_1 & a_2 - b_1 k_2 \\ a_3 - b_2 k_1 & a_4 - b_2 k_2 \end{bmatrix}$$

$$\det(sI - (A - BK)) = s^2 + \underline{\left[(a_1 - b_1 k_1) + (a_3 - b_2 k_2)\right]} s +$$

$$\underline{\left[(a_1 - b_1 k_1)(a_3 - b_2 k_2) + (a_2 - b_1 k_2)(a_3 - b_2 k_2)\right]} = \phi$$

2 eq'ns, 2 unknowns, — k_1 k_2

$x = Tz$

original — controller canonical form

$u = -\bar{k}T^{-1}x$

ccf

$K = [0 \; \text{———} \; 0 \;\; 1]\,\mathcal{C}^{-1}\alpha_c(A)$

$\mathcal{C} \triangleq [B \;\; AB \;\; A^2B \;\cdots\; A^{n-1}B]$ — controllability matrix

if Rank$(\mathcal{C}) = n$

put poles anywhere I want.

$\alpha_c(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I$

$\Delta_{cl} = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_n$

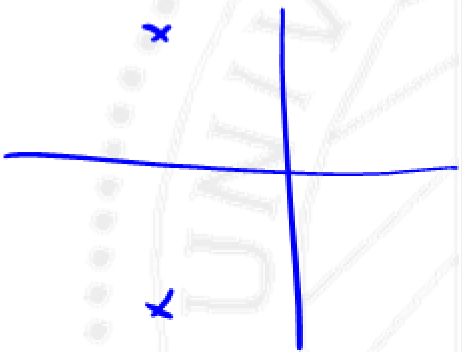
$$\mathcal{C} \triangleq \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

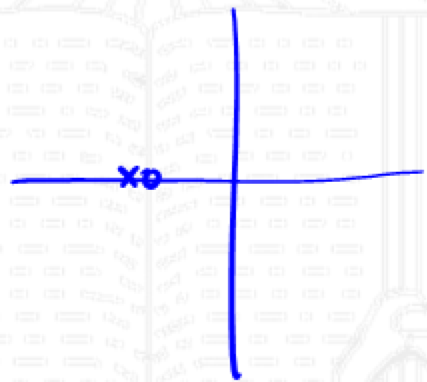
ctrb(A,B)

$\text{rank}(\mathcal{C}) = n$ for <u>control</u>.

Condition number of $\mathcal{C} \triangleq \dfrac{\sigma_{max}(\mathcal{C})}{\sigma_{min}(\mathcal{C})}$

<u>rcond</u>





Pole Placement

$$\dot{x} = Ax + Bu \qquad u = -Kx$$

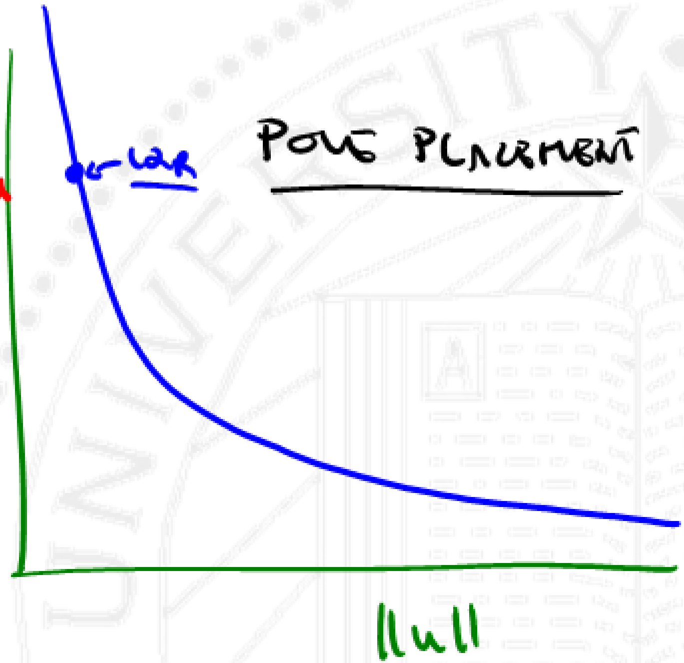
$$\mathcal{C} = [B \; AB \; \ldots \; A^{n-1}B]$$

$\text{rank}(\mathcal{C}) = n$ put poles arbitrarily

$\|x\|$

$\min \int_0^\infty (x^T Q x + u^T R u)\, dt$

subj $\dot{x} = Ax + Bu$



POLE PLACEMENT

State Space

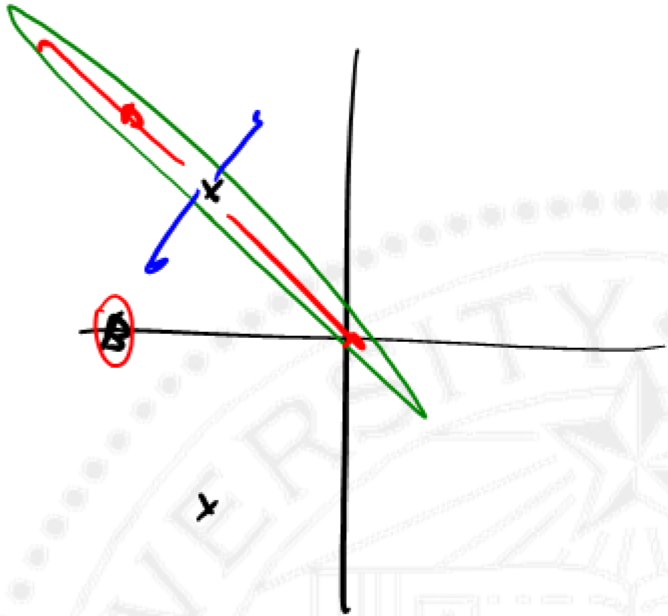
Control → w/ $u = -K\underline{x}$ | choose K to put $\mathrm{eig}(A-BK)$ where I want.

$\dot{x} = Ax + Bu$

$\mathrm{eig}(A-BK)$ where you want.

$\mathcal{C} = [B \; AB \; \dots \; A^{n-1}B]$ $\quad \underset{n}{\mathrm{rank}(\mathcal{C})}$, $\underset{\sim 1}{\mathrm{rcond}(\mathcal{C})}$.

FULL STATE FEEDBACK $u = -K\underline{x}$



MODAL COORDINATES

$$\dot{x} = Ax + Bu \longrightarrow x = Tz \;;\; z = T^{-1}x$$

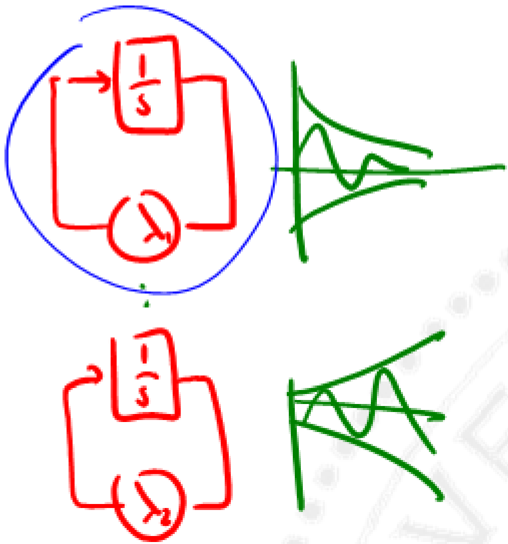
$$T^{-1}AT = \underbrace{\begin{bmatrix} (\;) & & 0 \\ & (\;) & \\ 0 & & \ddots \end{bmatrix}}_{\Sigma}$$

A block jordan form

$$\begin{bmatrix} -\zeta & \omega_n^2 \\ -\omega_n^2 & 1 \end{bmatrix}$$

$T^{-1}AT = \Lambda$ $\qquad$ $T^{-1}B$ — input

$$\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$



$$\dot{\underline{x}} = A\underline{x} + Bu \qquad u, \forall t \qquad x(0) = x_0$$

$$\underline{x}(t) = e^{At}\underline{x}_0$$

$$e^{At} \triangleq I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \quad \text{expm}$$

$$e^{\Sigma t} = \begin{bmatrix} e^{\Sigma_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\Sigma_n t} \end{bmatrix}$$

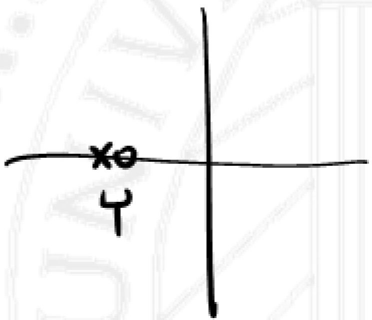
$$\mathcal{C} \triangleq \left[B \;\; AB \;\; \cdots \;\; A^{n-1}B \right]$$

$\mathrm{Rank}(\mathcal{C}) = n$ system is "controllable"

$$\kappa(\mathcal{C}) \triangleq \frac{\sigma_{max}}{\sigma_{min}}$$ ← singular values of $\mathcal{C}$

how close to singular you are

$\kappa(\mathcal{C})$ $1 \to \infty$.



$$\dot{x} = Ax + Bu \qquad x = Tz$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu$$

$$\mathcal{C} = \left[T^{-1}B \quad T^{-1}AT\underbrace{T^{-1}}B \quad T^{-1}A\underbrace{TT^{-1}}A\underbrace{TT^{-1}}B \quad \cdots \quad T^{-1}ATT^{-1}AT \cdots T^{-1}AT \right.$$

$$T^{-1}B \qquad T^{-1}A^{2}B \qquad T^{-1}A^{n-1}B$$

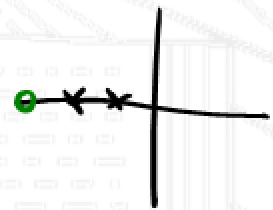
$$\underset{\text{new}}{\mathcal{C}} = T^{-1}\mathcal{C}_{old}$$

$$\dot{\underline{x}} = \begin{bmatrix}-3 & -2\\ 1 & 0\end{bmatrix}\underline{x} + \begin{bmatrix} \ \\ \ \end{bmatrix}\underline{u} \qquad y = [0 \ \ 1]\underline{x} + [0]u$$

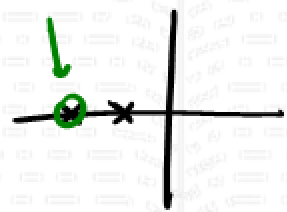
$$\mathrm{I}: \ B = \begin{bmatrix}0\\ 1\end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix}0 & -2\\ 1 & 0\end{bmatrix} \qquad \begin{aligned}\text{rank}(\mathcal{C}) &= 2\\ \text{ord}(\mathcal{C}) &= 2.\end{aligned}$$

$$C[sI-A]^{-1}B \longrightarrow [0 \ \ 1]\begin{bmatrix}s+3 & 2\\ -1 & s\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{s+3}{(s+2)(s+1)}$$

$$\mathrm{II}: \ B = \begin{bmatrix}-1\\ 1\end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix} \qquad \begin{aligned}\text{rank}(\mathcal{C}) &= 1.\\ \text{ord}(\mathcal{C}) &= 2.\end{aligned}$$



$$C[sI-A]^{-1}B \longrightarrow \frac{\cancel{s+2}}{\cancel{(s+2)}(s+1)}$$



$$\text{cond} \triangleq \frac{\sigma_{max}}{\sigma_{min}} = \frac{2}{\varepsilon} = \alpha \qquad \text{rcond}$$

$$B : \begin{bmatrix} -0.999 \\ 1 \end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix} -0.999 & 0.997 \\ 1 & -0.999 \end{bmatrix} \qquad \text{rank}(\mathcal{C}) = 2.$$

$$\text{cond}(\mathcal{C}) = \underline{4000}$$

$$C[sI - A]^{-1}B \longrightarrow \frac{s + 2.001}{(s+2)(s+1)}$$