CMPE-242
Applied Feedback Control

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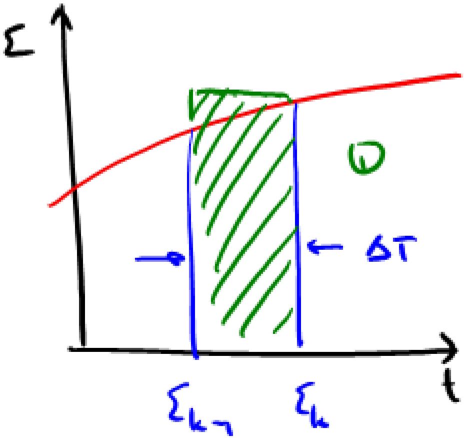
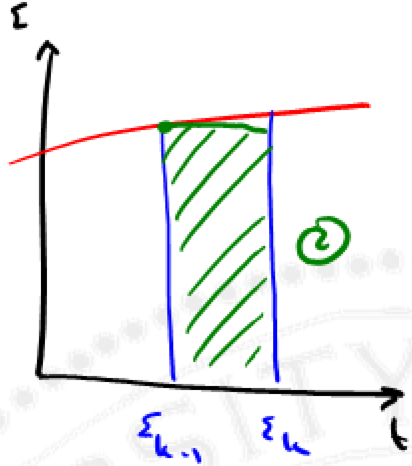
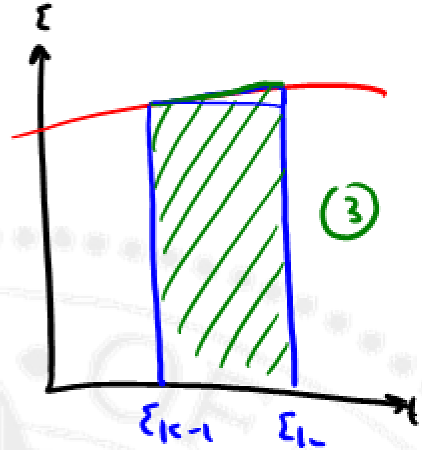
Winter 2016







$$A_k = A_{k-1} + E_k \Delta T$$

$$\boxed{s = \frac{z-1}{z\Delta T}} \quad ①$$

"Backwards Euler"

$$A_k = A_{k-1} + E_{k-1} \Delta T$$

$$(1-z^{-1})A_k = z^{-1}\Delta T E_k$$

$$\frac{A}{E} = \frac{z^{-1}\Delta T}{1-z^{-1}} = \frac{\Delta T}{z-1} = \frac{1}{s}$$

$$\boxed{s = \frac{z-1}{\Delta T}} \quad ②$$

"Forward"

$$A_k = A_{k-1} + \Delta T\left(\frac{E_k + E_{k-1}}{2}\right)$$

$$(1-z^{-1})A_k = \frac{\Delta T}{2}(1+z^{-1})E_k$$

$$\frac{A}{E} = \frac{\Delta T}{2}\frac{(1+z^{-1})}{(1-z^{-1})} = \frac{1}{s}$$

$$\boxed{s = \frac{2}{\Delta T}\frac{(z-1)}{(z+1)}} \quad ③$$

"Tustin"

1. $\frac{1}{s} = \frac{z\,\Delta T}{z-1} \longleftrightarrow s = \frac{z-1}{z\,\Delta T} \;\therefore\; z = \frac{1}{1-\Delta T s}$ "Backwards"

2. $\frac{1}{s} = \frac{\Delta T}{z-1} \longleftrightarrow s = \frac{z-1}{\Delta T} \;\therefore\; z = 1 + \Delta T s$ "Forwards"

3. $\frac{1}{s} = \frac{\Delta T}{2}\frac{(z+1)}{(z-1)} \longleftrightarrow s = \frac{2}{\Delta T}\frac{(z-1)}{(z+1)} \;\therefore\; z = \frac{1+\frac{\Delta T}{2}s}{1-\frac{\Delta T}{2}s}$ "Tustin"

4. $e^{-\Delta T s} = z^{-1} \longleftrightarrow s = \frac{1}{\Delta T}\ln(z) \;\therefore\; z = e^{\Delta T s}$ "Exact"

④ $z = e^{\Delta T \lambda} = \underline{1} + \underline{\Delta T \lambda} + \underline{\frac{\Delta T^2}{2!}} \lambda^2 + \frac{\Delta T^3}{3!} \lambda^3 + \cdots$

③ $z = \dfrac{1 + \frac{\Delta T}{2}\lambda}{1 - \frac{\Delta T}{2}\lambda} = \underline{1} + \underline{\Delta T \lambda} + \underline{\frac{\Delta T^2}{2}} \lambda^2 + \frac{\Delta T^3 \lambda^3}{4} + \cdots$

② $z = 1 + \Delta T \lambda = \underline{1} + \underline{\Delta T \lambda} + 0$

① $z = \dfrac{1}{1 - \Delta T \lambda} = \underline{1} + \underline{\Delta T \lambda} + \underline{\Delta T^2} \lambda^2 + \Delta T^3 \lambda^3 + \cdots$

$$K(s) = K\,\frac{s+a}{s+b}\Bigg|_{s=\frac{1}{\delta T}\ln(z)} = \frac{K\left(\frac{1}{\delta T}\ln(z)+a\right)}{\frac{1}{\delta T}\ln(z)+b}$$

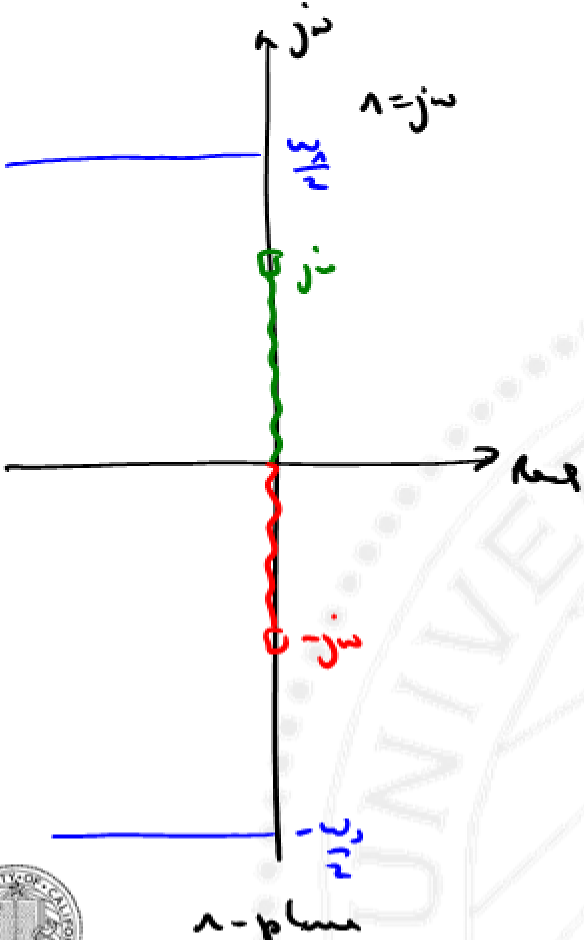
"hard to implement"

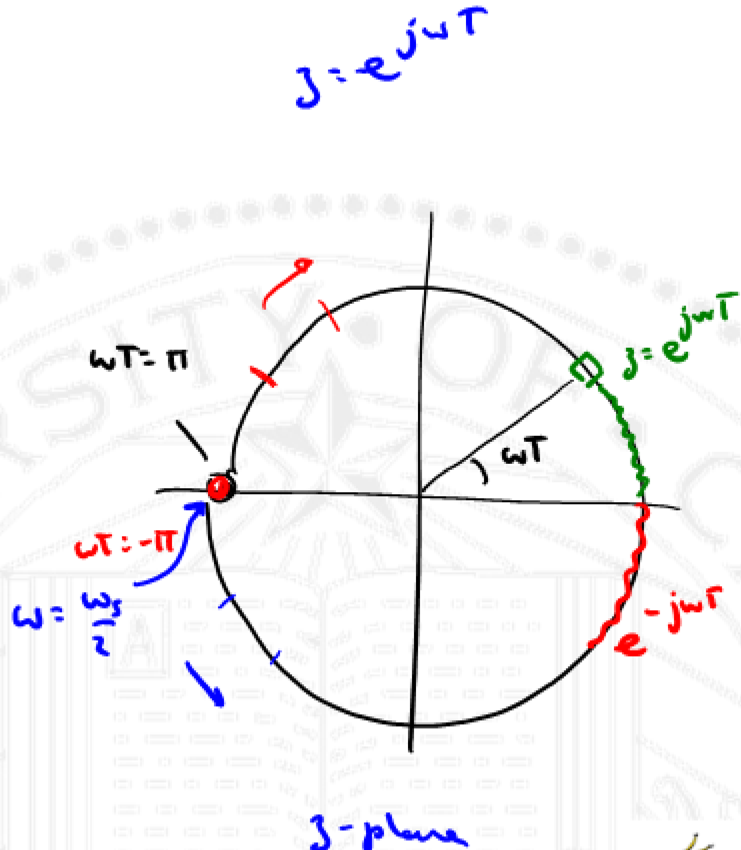
Exact is useful → equivalent poles are?

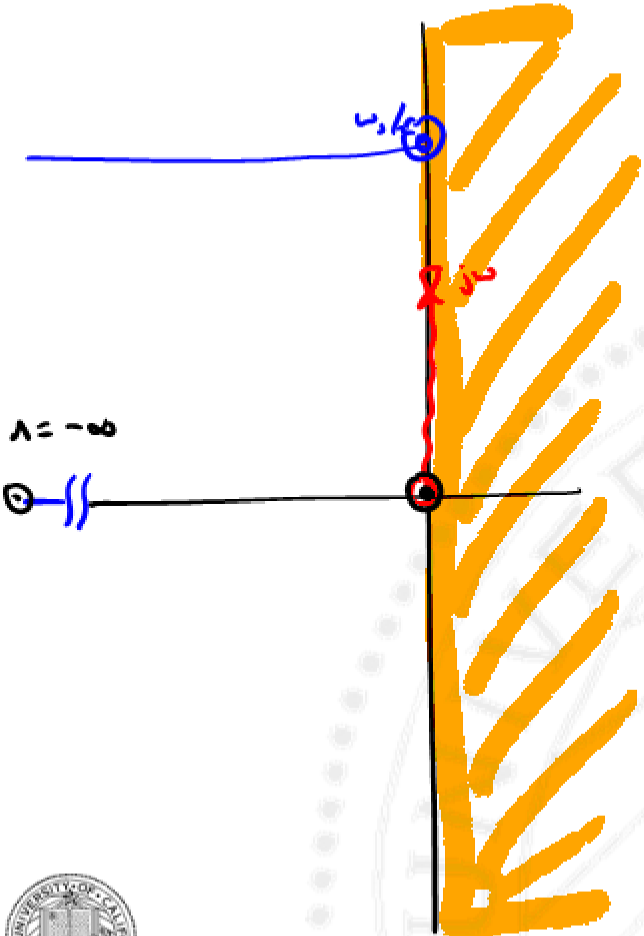
ODE → $K(s)$ $s = j\omega$ — Bode plot $\angle$, $\|$

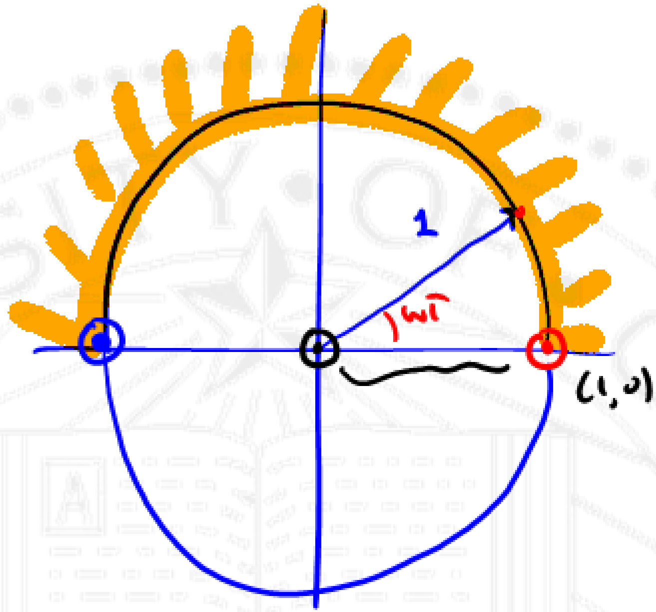
ODE → $K(z)$ $z = e^{\delta T s}$ — poles/zeros

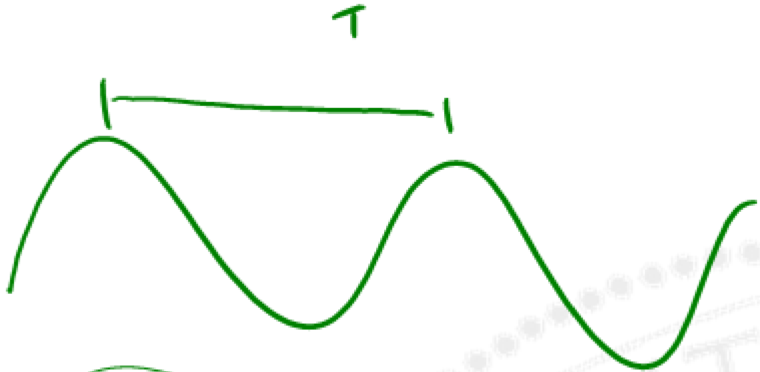
$K(z)\big|_{z=e^{j\omega\delta T}}$ — $\|$, $\angle$ (dBode)


jω
s = jω
ωs/2
jω
Re
-jω
-ωs/2
s-plane

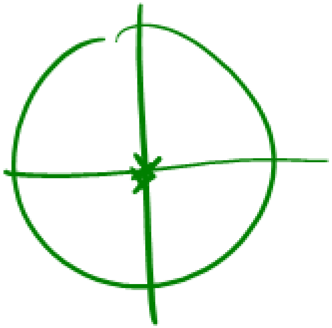

z = e^{jωT}
ωT = π
z = e^{jωT}
ωT
ωT = -π
ω = ωs/2
e^{-jωT}
z-plane


λ = -∞


1
(1,0)



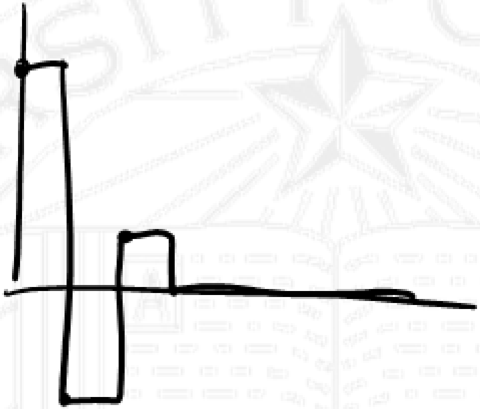
$$\omega_s = \frac{2\pi}{T} \qquad \omega = \frac{\omega_s}{T} \text{ Alias Freq.}$$



"Deadbeat Control"

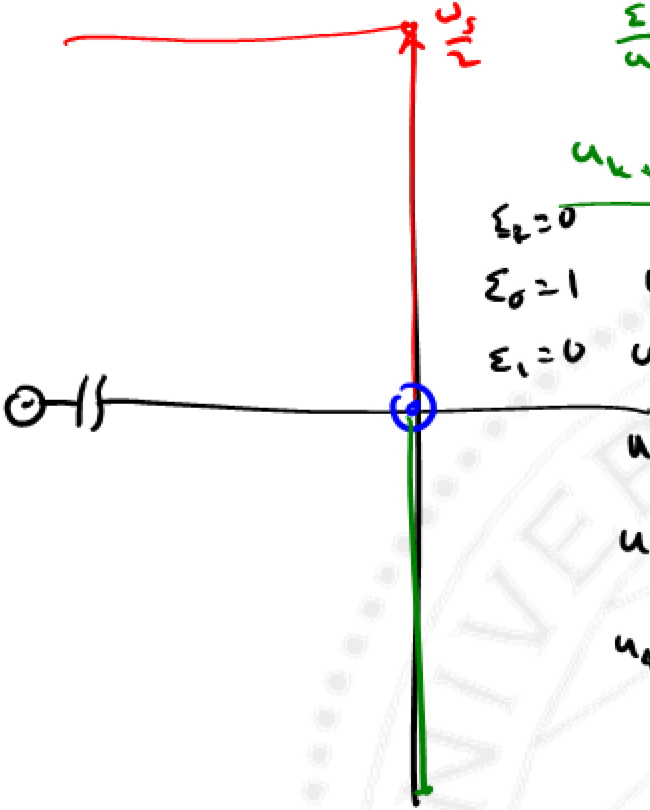
n^{th} order plant will go to zero in

n steps



$$\frac{E}{U} = z + \frac{1}{2} \rightarrow E_k = z u_k + \frac{1}{2} u_k$$

$$E_k = u_{k+1} + \frac{1}{2} u_k$$



$$u_{k+1} = E_k - \frac{1}{2} u_k$$

$E_{-1} = 0$

$E_0 = 1 \quad u_0 = 0$

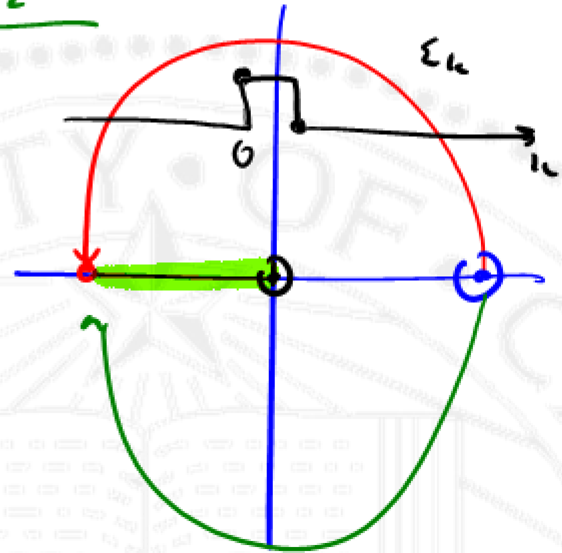
$E_1 = 0 \quad u_1 = 1$

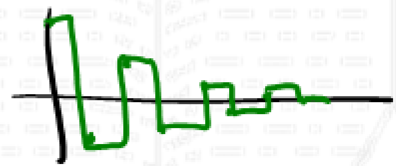
$u_2 = -\frac{1}{2}$

$u_3 = +\frac{1}{4}$

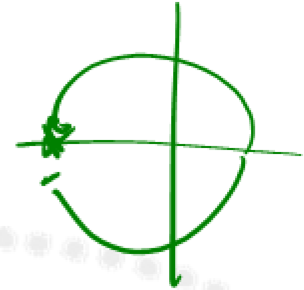
$u_4 = -\frac{1}{8}$

$\vdots$



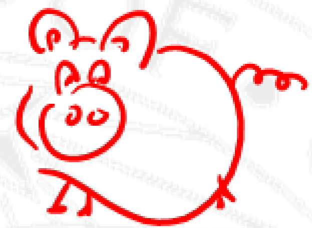


$\lambda = \frac{1}{T}\ln(z) \qquad z = -\frac{1}{2} \qquad z = \frac{1}{2}e^{\pm\pi j}$



$$\lambda = \frac{1}{T}\ln\left(\frac{1}{2}e^{\pm\pi j}\right) = \frac{1}{T}\ln\left(\frac{1}{2}\right)\ln\left(e^{\pm\pi j}\right)$$

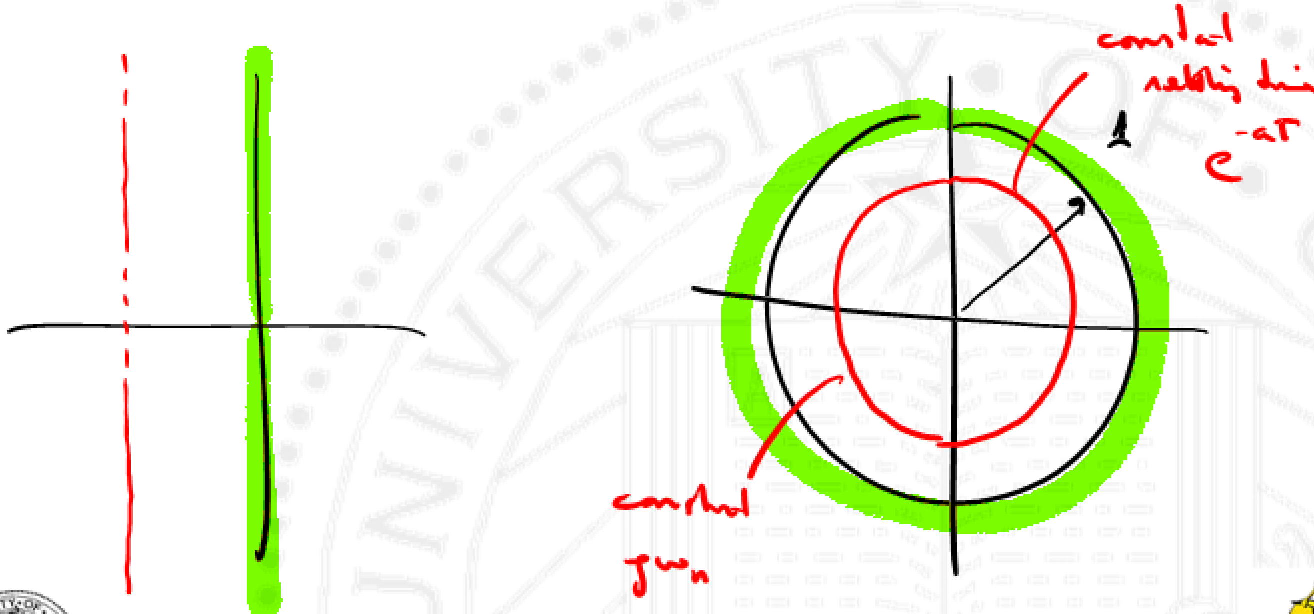
$$\boxed{= \frac{1}{T}\ln\left(\frac{1}{2}\right) \pm \pi j}$$



not kosher

$$s = \frac{1}{T}\ln(z) \longleftrightarrow z = e^{Ts}$$

$$s = -a + bj \longleftrightarrow z = e^{(-a+bj)T} = e^{-aT} e^{jbT}$$



STABILITY $\longleftrightarrow$ INSIDE UNIT CIRCLE

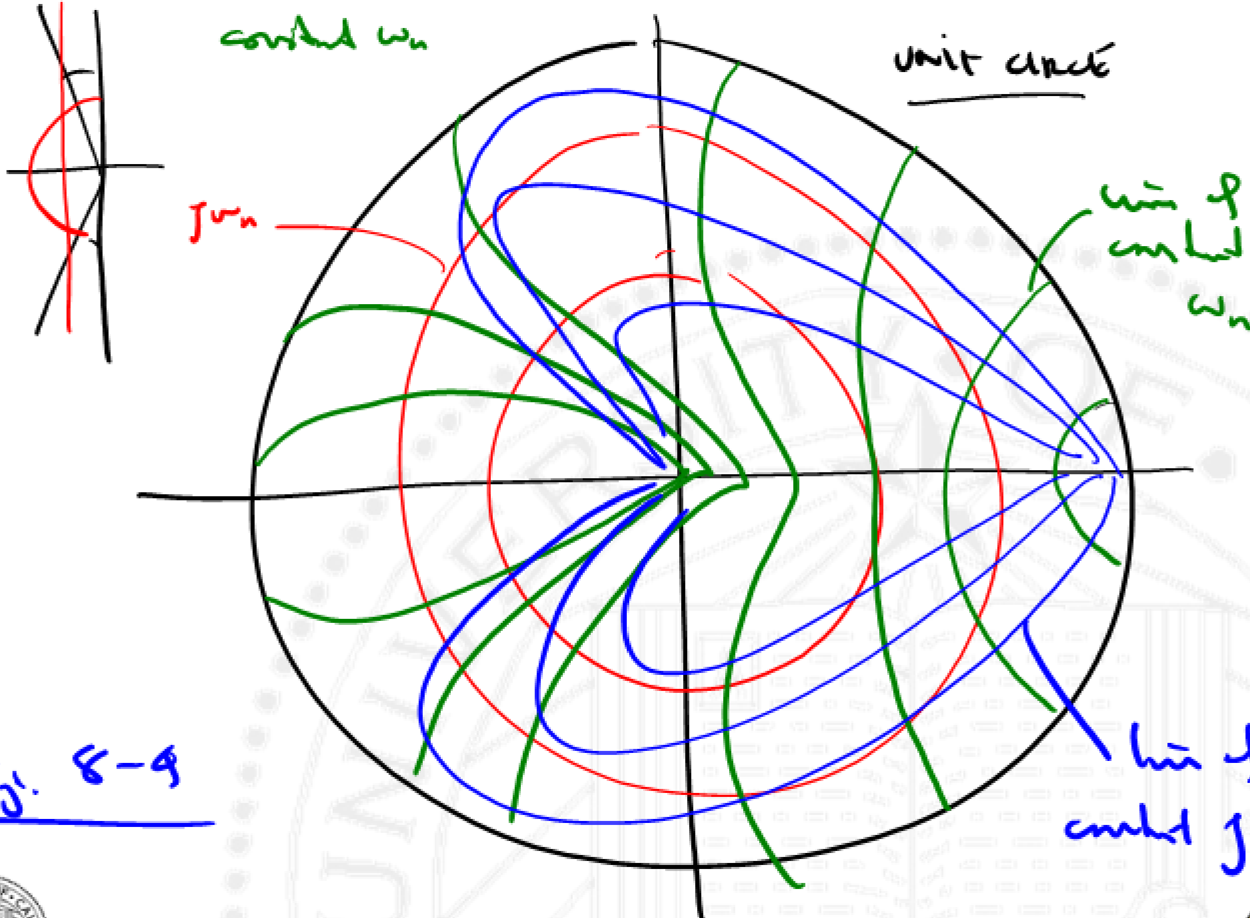
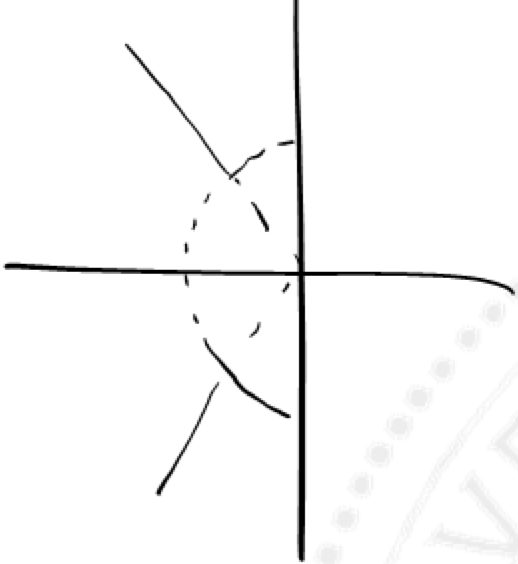
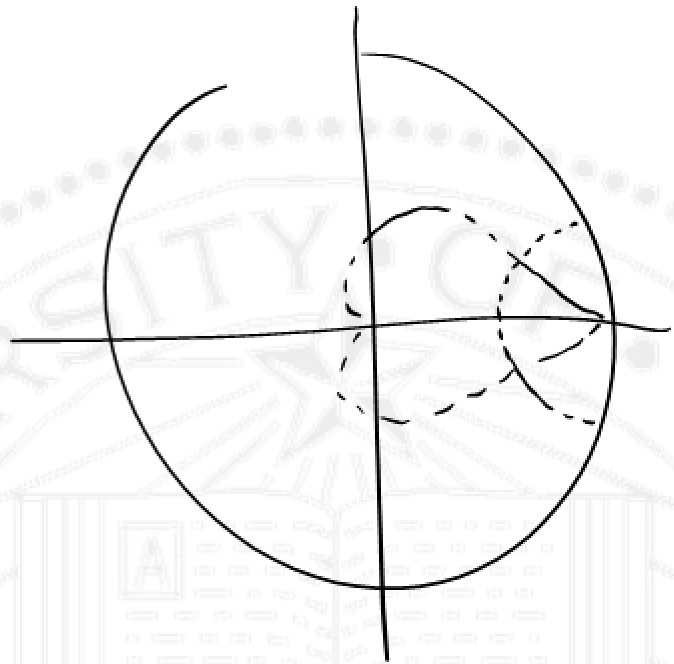

constant ωn
unit circle
ζωn
lines of constant ωn
lines of constant ζ

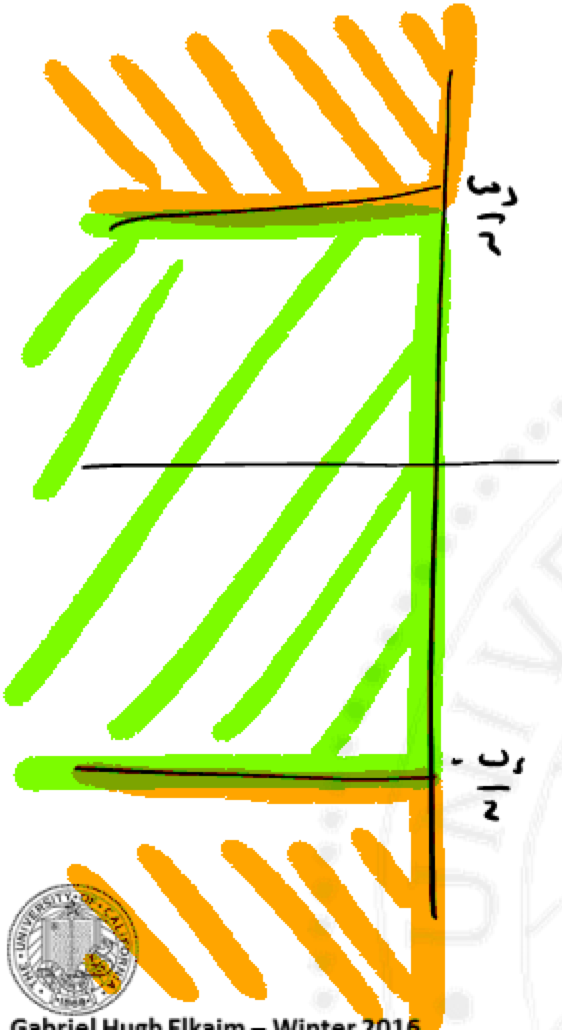
Fig. 8-9

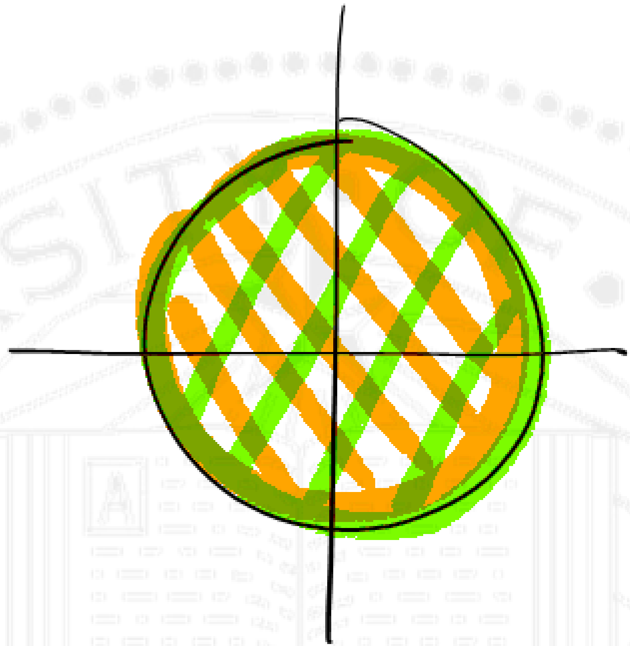
Sgrid(ω_n, ζ)

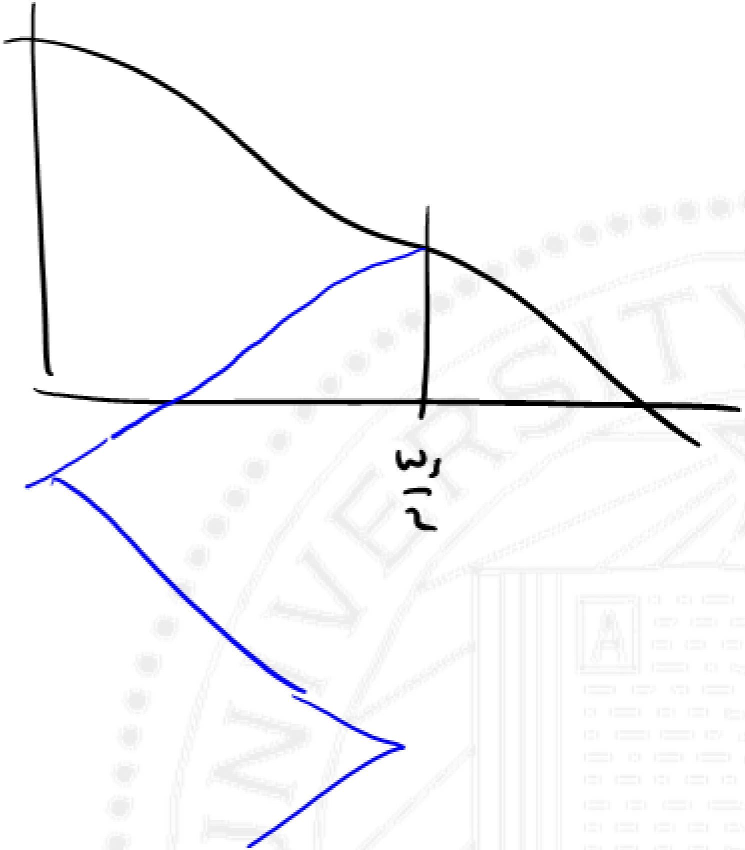


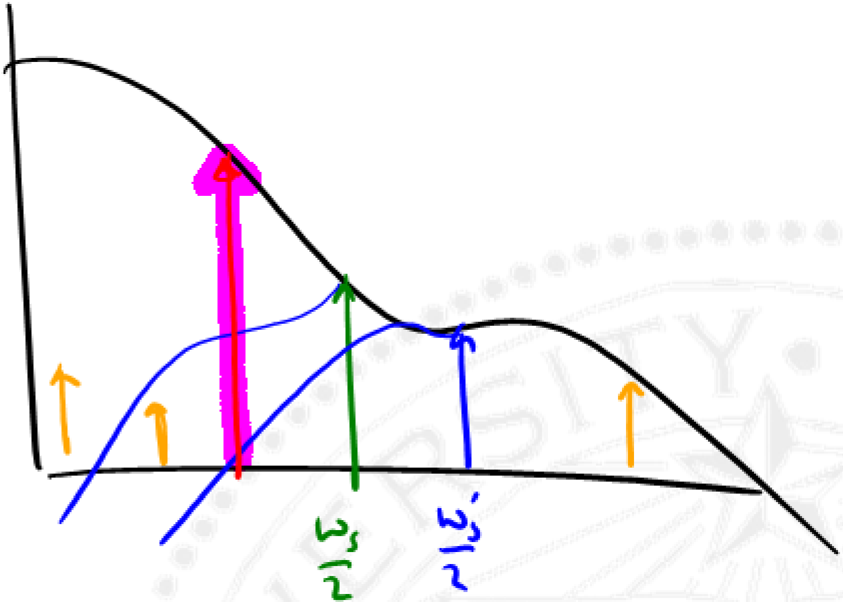
zgrid(ω_n, ζ)











~

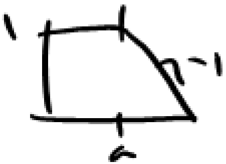
not multiples of some other freq.

Design in analog world ~ account for $\frac{\delta T}{2}$ delay

Convert $K(s) \longrightarrow K(z) \Big|_{\text{Tustin}}$

$s = \frac{1}{\delta T} \ln(z)$

$$K(s) = \frac{a}{s+a}$$



$$K(z) = \left.\frac{a}{s+a}\right|_{s=\frac{z-1}{T}} = \frac{a}{\frac{z-1}{T}+a} = \frac{aT}{z-1-aT} = \frac{aT}{z-(1-aT)}$$ "Forwards"

$$K(z) = \left.\frac{a}{s+a}\right|_{s=\frac{z-1}{Tz}} = \frac{a}{\frac{z-1}{Tz}+a} = \left[\frac{aT}{1+aT}\right]\frac{z}{z-\left(\frac{1}{1+aT}\right)}$$ "Backwards"

$$K(z) = \left.\frac{a}{s+a}\right|_{s=\frac{2}{T}\frac{z-1}{z+1}} = \frac{a}{\frac{2}{T}\frac{z-1}{z+1}+a} = \left[\frac{\frac{aT}{2}}{1+\frac{aT}{2}}\right]\frac{z+1}{z-\left(\frac{1-aT/2}{1+aT/2}\right)}$$

DIGITAL CONTROL VIA MAPPING

(1) Account for $\frac{\Delta T}{2}$ delay < Bode ~ $\Delta\phi = -\frac{\omega T}{2}$; RL ~ Pade (0°)

(2) Sample @ 20-30x ω_{xo} (ω_n)

(3) Design $K(s)$ using analog tools

(4) $K(z) \Leftarrow K(s)$ using TUSTIN

(5) Specific freq of interest (ω_{xo}) → Prewarp

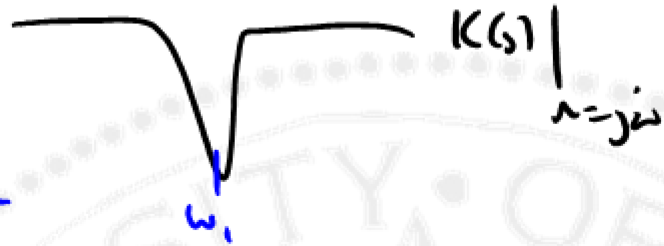
Prewarping

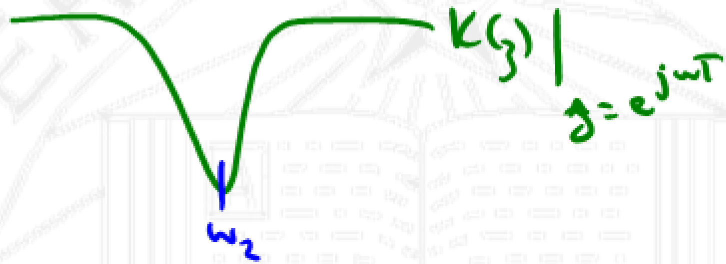
$K(s)$ — "notch"

$\Downarrow$ Tustin

$K(z)$

$\omega_1 \neq \omega_2$





$$K(s) \longrightarrow K(z)$$

$$s = j\omega_{an} = \frac{2}{T}\frac{e^{j\omega_1 T}-1}{e^{j\omega_1 T}+1}$$

$$e^{j\omega_1 T} = \cos(\omega_1 T) + j\sin(\omega_1 T)$$

$$j\omega_{ans} = \frac{2}{T}\tan\left(\frac{\omega_1 T}{2}\right)$$

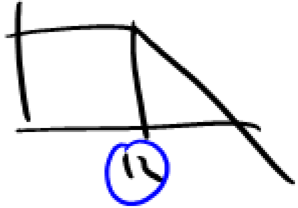
prewarping

$$K(z) = K(s)\Bigg|_{s = \left[\frac{\omega_1}{\tan\left(\frac{\omega_1 T}{2}\right)}\right]\frac{z-1}{z+1}}$$

Tustin w/ Prewarping

$$s = \left[\frac{\omega_1}{\tan\left(\frac{\omega_1 T}{2}\right)} \right] \frac{z-1}{z+1}$$

$K(s) = \frac{12}{s+12}$



25 Hz: $T = \frac{1}{25}$

$$K(z) = K(s)\Big|_{s = \left(\frac{\omega_1}{\tan\left(\frac{\omega_1 T}{2}\right)}\right)\frac{z-1}{z+1}} = \frac{12}{s+12}\Big|_{\left(\frac{12}{\tan\left(\frac{12/25}{2}\right)}\right)\frac{z-1}{z+1}}$$

$$K(z) = \frac{12}{\frac{0.244}{}\frac{z-1}{z+1} + 12}$$

$$\frac{T}{2} \sim \frac{1}{50} - \underline{0.02}$$

Z-transform

$$\mathcal{L}\{f(t)\} \triangleq F(s) = \int_0^\infty f(t)e^{-st}\,dt \begin{cases} \text{conv} \to x \\ \dot{x} \to \text{n + i.c.'s} \end{cases}$$

$$\mathcal{Z}\{f(kT)\} \triangleq F(z) = \sum_0^\infty f_k z^{-k} \begin{cases} \text{conv} \to x \\ x_{k-1} = z^{-1}x_k \,. \end{cases}$$

$f(kT) \uparrow f_k$

impulse $\mathcal{L}\{\delta\} = 1$.

unit pulse $Z\{\cdot\} = \sum_0^\infty f_k z^{-k} = 1z^0 + 0z^{-1} + 0z^{-2} + \ldots$

$= 1$.

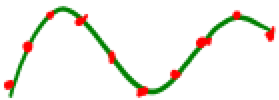
step $\mathcal{L}\{\cdot\} = \frac{1}{s}$

$Z\{\cdot\} = 1z^0 + 1z^{-1} + 1z^{-2} + \ldots = \sum_0^\infty z^{-k}$

$$\sum_0^\infty \alpha^k = \frac{1}{1-\alpha} \quad \forall\ |\alpha| < 1$$

$$\alpha = z^{-1} \qquad \sum_0^\infty z^{-k} = \frac{1}{1-z^{-1}} = \boxed{\left(\frac{z}{z-1}\right) = Z(\cdot)}$$

$f(t)$ $\mathcal{L}\{f(t)\}$ $\mathcal{Z}\{f(kT)\}$



Inverse $\mathcal{Z}$-transform

$\mathcal{L}^{-1}\{\} \rightarrow f(t)$

$\mathcal{Z}^{-1}\{\} \rightarrow f_k$

$f(kT)$

$1 + 1z^{-1} + 1z^{-2} \cdots$

$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} \longrightarrow$

$$\begin{array}{r|l} z-1 & z \\ & z-1 \\ \hline & +1 \\ & 1-z^{-1} \\ \hline & z^{-1} \end{array}$$

// Prove that z^{-1} is a unit delay

$$\mathcal{Z}\{f_{k-1}\} = \sum_{k=0}^{\infty} f(k-1) z^{-k} \qquad \text{let } j = k-1$$

$$= \sum_{j=0}^{\infty} f(j) z^{-(j+1)} = \sum_{j=0}^{\infty} f(j) z^{-j} z^{-1}$$

$$= z^{-1} \underbrace{\sum_{j=0}^{\infty} f(j) z^{-j}}_{F(z)} = z^{-1} F(z)$$

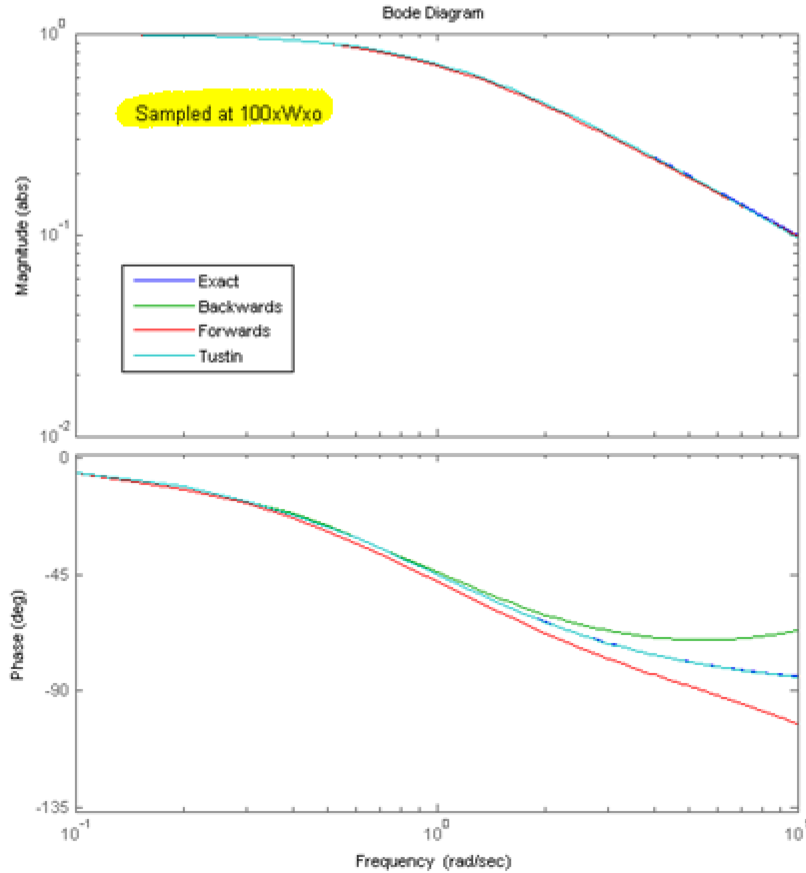
MATLAB CODE

```
w=[0.1:.1:10]';
T=2*pi/100;
a=1;
n=a;
d=[1 a];
n1=[a*T/(1+a*T) 0];
d1=[1 -1/(1+a*T)];
n2 = [0 a*T/(1+a*T)];
n3 = a*T/(2+a*T)*[1 1];
d3 = [1 -(1-a*T/2)/(1+a*T/2)];
bode(n,d,w); hold on
dbode(n1,d1,T,w);
dbode(n2,d1,T,w);
dbode(n3,d3,T,w);
legend('Exact','Backwards','Forwards','Tustin');
```

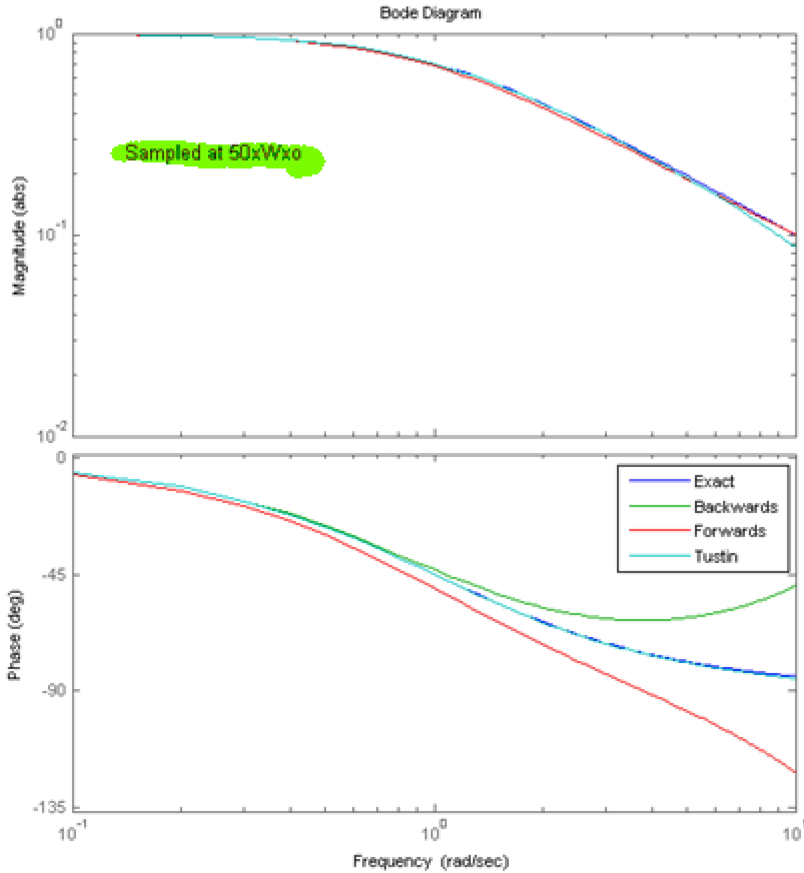
$\left(\frac{a}{s+a}\right)$

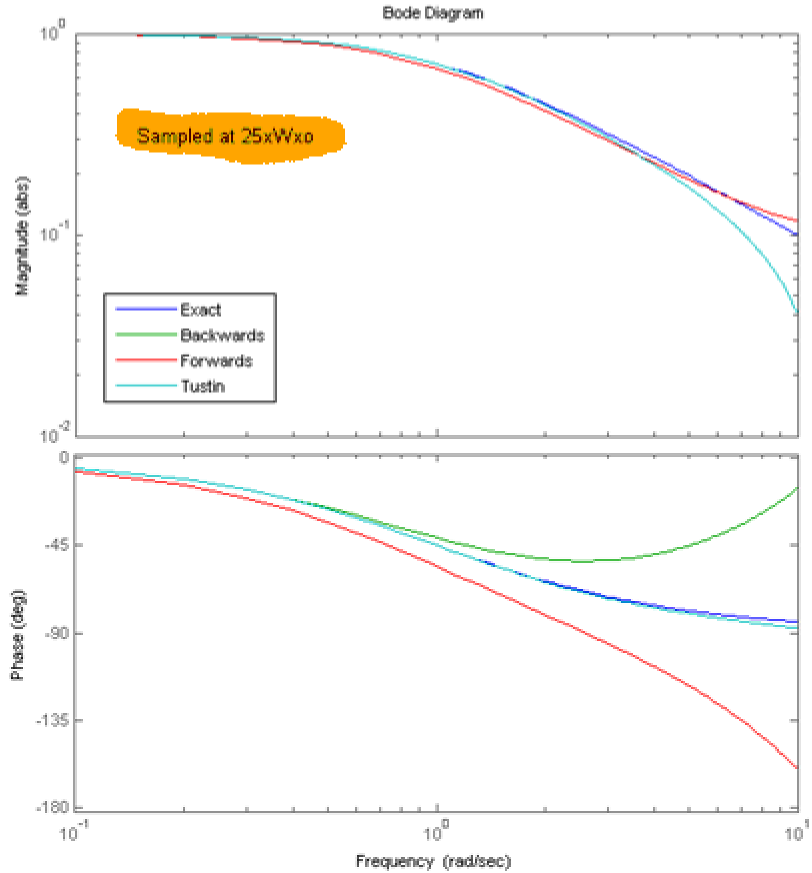
$\frac{z+2}{z^2+2z+5}$

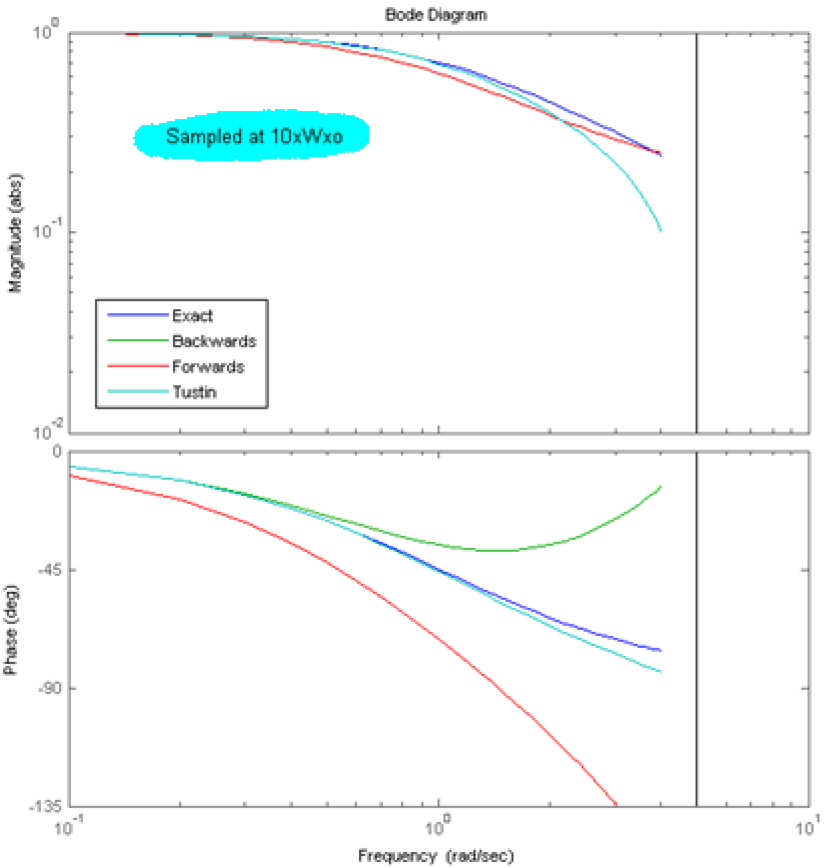
[0 1 2],[1 2 5]

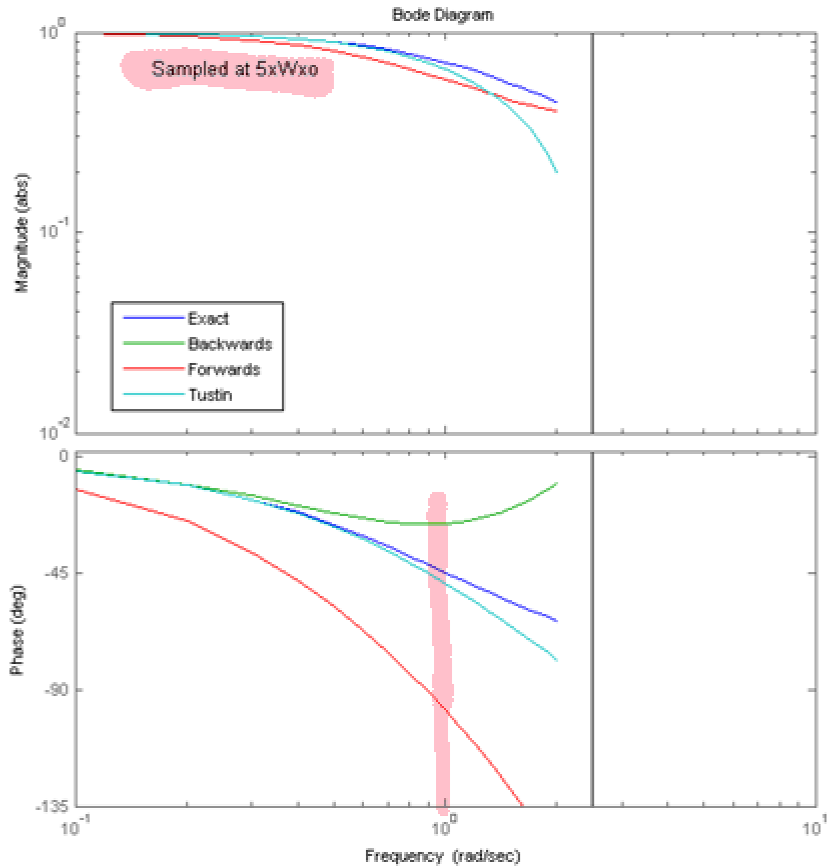


Bode Diagram
Sampled at 100xWxo
Magnitude (abs)
Exact
Backwards
Forwards
Tustin
Phase (deg)
Frequency (rad/sec)


Bode Diagram
Sampled at 50xWxo
Magnitude (abs)
Phase (deg)
Frequency (rad/sec)
Exact
Backwards
Forwards
Tustin


Bode Diagram
Sampled at 25xWxo
Magnitude (abs)
Phase (deg)
Frequency (rad/sec)
Exact
Backwards
Forwards
Tustin
10^0
10^{-1}
10^{-2}
0
-45
-90
-135
-180
10^{-1}
10^0
10^1


Bode Diagram
Sampled at 10xWxo
Magnitude (abs)
Phase (deg)
Exact
Backwards
Forwards
Tustin
Frequency (rad/sec)


Bode Diagram
Sampled at 5xWxo
Magnitude (abs)
Exact
Backwards
Forwards
Tustin
Phase (deg)
0
-45
-90
-135
Frequency (rad/sec)