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# DESIGN OF CASCADE COMPENSATORS USING BODE PLOT AND ROOT LOCUS METHODS

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These notes describe some procedures for designing cascade compensators using Bode plot and root locus methods. These two approaches should not be viewed as competitive but as complementary. With the root locus or s-plane design, one is able to produce a transient response that is desirable by choosing dominant closed loop poles, but meeting error constant specifications is not as apparent. Frequency response methods easily treat error constants and bandwidth specifications, but controlling the transient response leaves something to be desired.

Throughout the notes we assume that the compensator is of the general form

$$G_c(S) = K \frac{((s/a) + 1)}{((s/b) + 1)} = K \frac{b}{a} \frac{s + a}{s + b} \qquad (1)$$

In the first expression, the multiplying constant of this and other transfer functions in this form is called the Bode gain (K), the d.c. gain excluding integration; this is the standard form before drawing Bode plots. The second expression shows the root locus gain (K b/a) since this is the standard form before drawing root loci.

If a, the frequency of the zero, is less than b, the frequency of the pole, the compensator is called a lead compensator because the net phase angle is nonnegative at all frequencies; when b is less than a, we have a lag compensator. Generally speaking, cascade lead compensators are used to increa bandwidth and thus decrease the response time of the compensated system. Lag compensators may make the system more sluggish and they are used to increase the low frequency gain (error constant) of the system.

Many students of automatic control have felt that the topic of compensation was a black art which could be mastered only after ten or fifteen years of hard work. We have tried to overcome this not by presenting different networks and showing what they can do, but by solving realistic design situations; for example: "Meet bandwidth and phase margin specs with an unspecified change in gain (error constant)". Each situation has a set of step-by-step instructions to calculate the compensator parameters K, a, and b. Obviously these solutions are not all inclusive, nor were they intended to be, but they provide a good starting point for the inexperienced designer to approach other design problems.

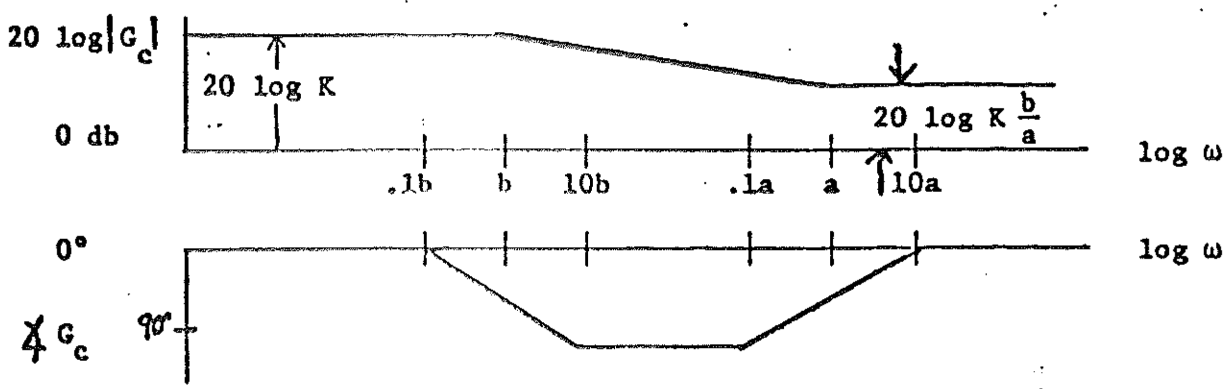
## DESIGN OF CASCADE COMPENSATORS USING BODE PLOTS

The crossover frequency $\omega_c$, (the frequency where the magnitude of the open-loop frequency response is unity) will be taken as a measure of the closed loop bandwidth (-3 db point). Examination of a Nichols chart shows that these are equal if the open-loop phase happens to be -90° at this frequency. The Nichols chart also shows that the closed-loop amplitude at $\omega_c$ increases for decreasing phase at $\omega_c$. Thus the actual closed-loop bandwidth will be less than $\omega_c$ if the open-loop phase lag is greater than 90°.

### I. To Meet an Error Constant Spec but Keep Phase Margin and Bandwidth Constant

This situation arises when the stability characteristics and response time are sufficient, but the system exhibits large steady state errors to step ( a Type 0 system) or ramp (Type 1) inputs. This calls for an increase in the Bode gain, but doing so without compensation will raise the crossover frequency and decrease the phase margin in most cases. The idea is to choose a compensator which does not disturb the frequency response in the vicinity of the crossover (and thus change the stability characteristics of the uncompensated system), while allowing an increase in Bode gain (to increase the error constant).

The compensator that meets these requirements is a lag compensator, so called because it has phase lag at all frequencies. In terms of the form shown in (1) we have b < a for a lag compensator. The straight-line Bode plots for the lag compensator are sketched below.




The rationale behind the design of lag compensators is to choose the frequencies b and a at least a decade below the crossover frequency, so that the compensator will contribute less than 6° lag at $\omega_c$. To maintain the same crossover, choose a/b equal to K, where K is the desired increase in the error constant. Now the lag compensator has a gain of unity and essentially no phase lag at the crossover frequency. We have met our objectives of increasing the gain yet leaving the frequency response unchanged near crossover. A summary of the design procedure is given below.

*TABLE I. TO MEET AN ERROR CONSTANT SPEC BUT KEEP PHASE MARGIN AND BANDWIDTH CONSTANT*

1. *Determine the required increase in Bode gain and set the compensator gain K to this value.*

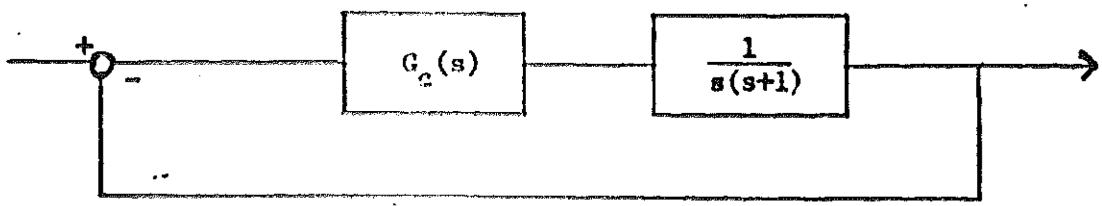
2. *Let the frequency ratio $\frac{a}{b}$ equal K.*

3. *Choose the zero frequency a at least a decade below the crossover frequency to avoid phase lag contributions at $\omega_c$.*

4. *The frequencies a and b are found from steps 2 and 3.*

Note that we are using the attenuation characteristics of the lag compensator which we counteract by raising the Bode gain. The effect of its phase lag, which is destabilizing, is minimized.

**Example**




Specifications:

Velocity constant $\geq 10\ sec^{-1}$

Phase margin $= 45° \pm 10°$

Crossover frequency = 1 rad/sec

The straightline Bode plots for the uncompensated system are shown in Figure 1. We see that the phase margin and bandwidth specs are satisfied, but the Bode gain (the velocity constant) is only 1 $sec^{-1}$ without compensation. A lag compensator will do the job, and its design can be found by the procedure outlined in Table I.

1. K = increase in open-loop gain = 10
2. a/b = 10
3. $\omega_c$ = 1 so a $\leq$ 0.1; choose a = 0.1
4. b = (b/a)a = (1/10)(0.1) = 0.01

$$G_c(s) = (10)\,\frac{1 + \frac{s}{0.1}}{1 + \frac{s}{.01}}$$

The dotted curve in Figure 1 is the straightline Bode plot for the compensated system. Observe that the frequency response is unchanged (within the straightline approximation) in the vicinity of crossover. Actually, the phase margin is decreased by approximately 6° so that tighter tolerances on its specification would mean that a, the zero frequency, would have to be lower than $\omega_c$.

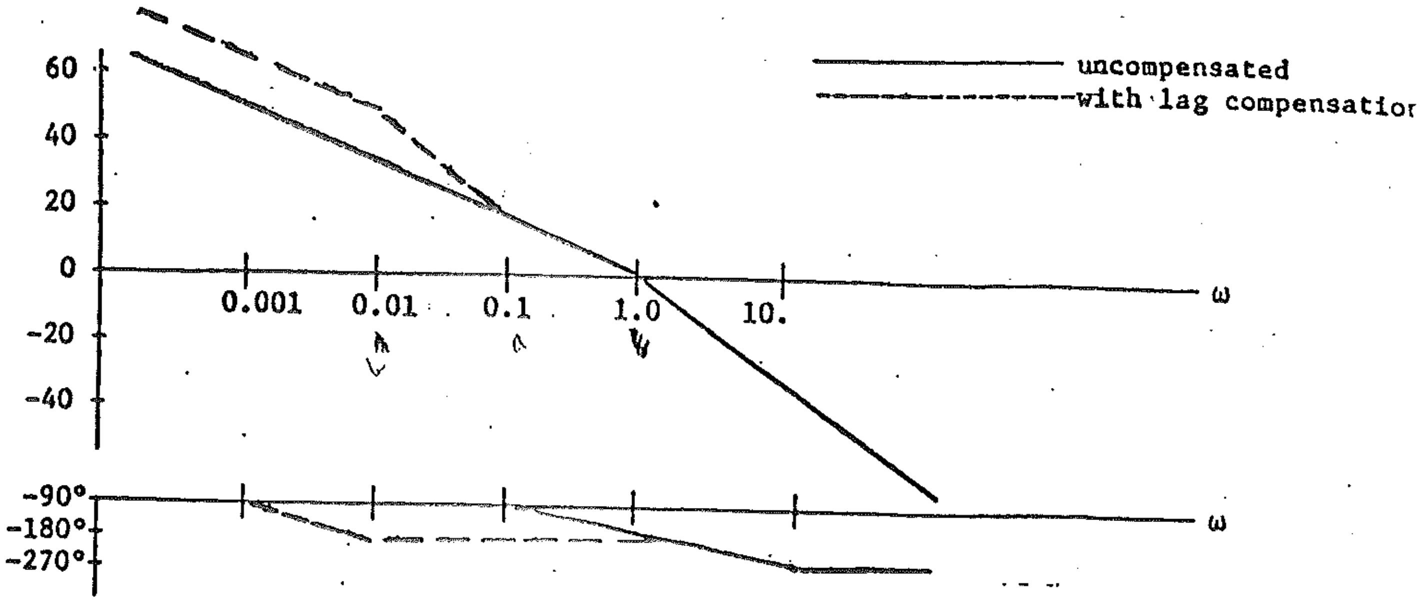



Figure 1. Lag compensation of 1/s(s+1)

## II. To Meet Bandwidth and Phase Margin Specs with an Unspecified Change in Gain (Error Constant)

In this situation it is desired to have a specific crossover frequency and phase margin. We shall discuss the more prevalent situation of increasing the bandwidth (crossover frequency) to improve the system response time, although an analogous procedure can be used to decrease the bandwidth. There is no theoretical limitation to the increase in bandwidth that can be obtained, but a practical limit is set by the saturation levels of the various components. The final design should be checked to see that these levels are not exceeded.

The crossover frequency can be made larger by increasing the gain alone, but in many instances the phase margin is too small at the new frequency. The latter difficulty can be overcome by inserting a lead network (a < b in (1)) to add phase lead to the open-loop frequency response. Here we are using the phase characteristic of the compensator, whereas with the lag network we use the attenuation or amplitude characteristic. Some important parameters of the lead network are summarized below:

$$G_c(s) = K\frac{1+\frac{s}{a}}{1+\frac{s}{b}} \qquad a < b \tag{2}$$

$$\text{d.c. gain} = K \tag{3}$$

$$\text{high frequency gain} = K\frac{b}{a} > K \tag{4}$$

$$\text{maximum phase lead, } \phi_{max}: \sin\phi_{max} = \frac{\frac{b}{a}-1}{\frac{b}{a}+1} \tag{5}$$

$$\text{frequency for maximum lead, } \omega_{max}: \omega^2_{max} = ab \tag{6}$$

$$\text{gain of } \frac{1+\frac{s}{a}}{1+\frac{s}{b}} \text{ at } \omega = \omega_{max} \quad 20\log|G(j\omega_{max})| = 10\log\frac{b}{a} \tag{7}$$

As a practical matter, the lead ratio b/a is usually not made larger that about 10 or 15 corresponding to maximum lead angles of 55° and 61°, respectively. Not only do the networks become harder to build, but there is very little increase in $\phi_{max}$ for corresponding increases in b/a beyond b/a = 15. (See Figure 10-5, p. 308 in Dorf). Cascading two or more lead networks can overcome this problem.

The design procedure to meet bandwidth and phase margin specs is straightforward and is summarized below.

*TABLE II. <u>TO MEET BANDWIDTH AND PHASE MARGIN SPECS WITH AN UNSEPECIFIED CHANGE IN GAIN (ERROR CONSTANT)</u>*

1. *Draw the open-loop Bode plots of the uncompensated system.*

2. *Determine the phase lead required at the specified crossover frequency to meet the phase margin spec.*

3. *Let this required lead angle be $\phi_{max}$ and solve (5) for b/a.*

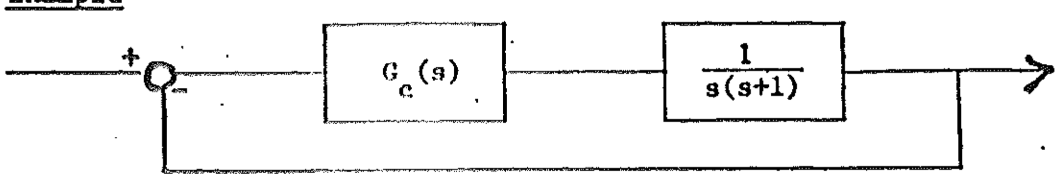
4. *Let this maximum lead occur at $\omega_c$ so that (6) becomes $\omega_c^2 = ab$. Use this equation and the result of step 3 to find a and b.*

5. *Graphically add the response of the lead network to the amplitude portion of the Bode plot of the uncompensated system.*

6. *Adjust the gain of the compensator K until the amplitude of the compensated system goes through 0 db at $\omega = \omega_c$.*

Although crossover frequency and phase margin are set to prescribed values, there is no flexibility in the choice of the compensator gain K. This is illustrated in the following example.

<u>Example</u>




Specifications: $\omega_c$ = 10 rad/sec  phase margin = 40°

The procedure of Table II is outlined below. References are made to curves in Figure 2 which are labeled with the step number.

1. Straightline Bode plots of the uncompensated system are labeled 1.
2. 40° phase lead required at $\omega_c$ = 10 rad/sec to have phase margin of 40°.

3. From (5)

$$\sin\phi_{max} = \sin 40^\circ = \frac{\frac{b}{a} - 1}{\frac{b}{a} + 1} \quad \text{OR} \quad \frac{b}{a} = \frac{1 + \sin\phi_{max}}{1 - \sin\phi_{max}} = \frac{1.64}{0.36} = 4.55$$

4. $\omega^2_{max} = \omega^2_c = 10^2 = ab \qquad b^2 = (\frac{b}{a})ab = (4.55)100 \qquad b = 21.4$

$$a = b(\frac{a}{b}) = \frac{21.4}{4.55} = 4.7$$

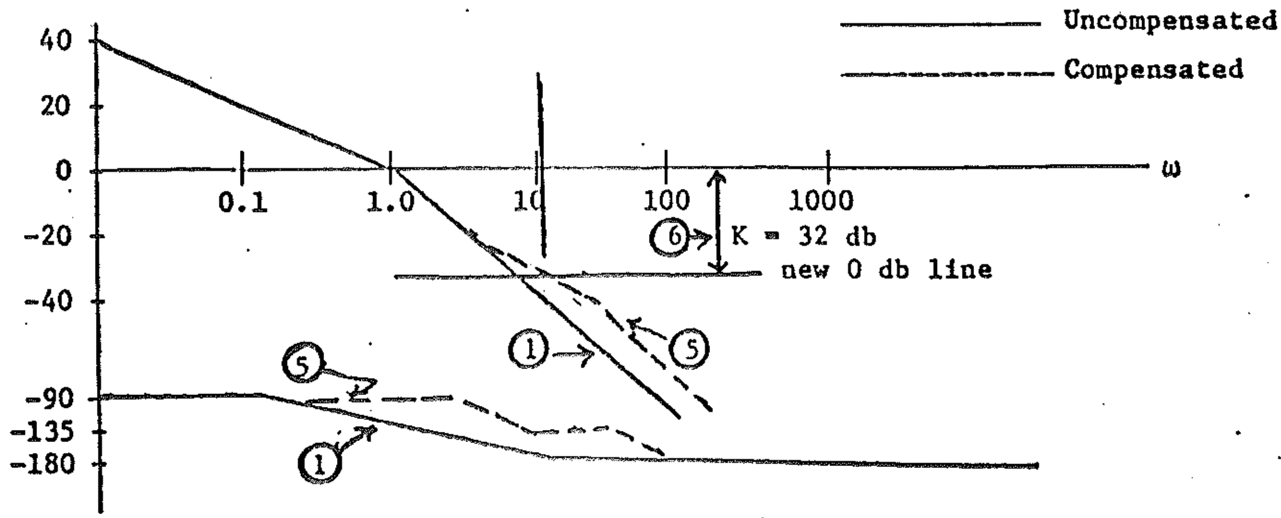



Figure 2. Lead compensation of 1/s(s+1)

5. The curve (5) is found by changing the slope of the uncompensated system by +1 (20 db/dec) at $\omega = 4.7$, and by -1 (-20 db/dec) at $\omega = 21.4$. The modified curves for phase are also shown but are not needed in the design procedure.

6. The compensator gain K is chosen to make the compensated plot go through 0 db at $\omega_c$ (10 rad/sec). This can be done by <u>raising the amplitude plot</u> or <u>by lowering the 0 db line</u>. The latter method is easier, and the amount of gain required is shown as line (6) on the plot. From this we see:

$$20 \log K = 32 \text{ db}, \quad K = 40$$

The compensator for this problem is

$$G_c(s) = 40 \; \frac{1 + \frac{s}{4.7}}{1 + \frac{s}{21.4}}$$

Although the Bode gain was increased in this case, it would have been reduced had the bode plot of the compensated system been above 0 db at $\omega_c$.

## III. To Meet Error Constant and Phase Margin Specs with an Unspecified Change in Bandwidth

This situation should be compared with the last one: here we design to specified phase margin and error constant (with an unspecified change in bandwidth) as opposed to a design to specified phase margin and bandwidth (with unspecified error constant). This design procedure is an iterative one, and to best explain it, we first summarize the steps and then expand on them through an example.

*TABLE III. TO MEET ERROR CONSTANT AND PHASE MARGIN SPECS WITH AN UNSPECIFIED CHANGE IN BANDWIDTH*

1. *Draw the Bode plots of the uncompensated system.*

2. *Choose the compensator gain K to meet the error constant requirement and draw the new 0 db line corresponding to this gain-compensated system.*

3. *Find the phase lead required at the crossover frequency of the gain-compensated system. Add 10% (but see the example) and let the total be $\phi_{max}$ of the compensator.*

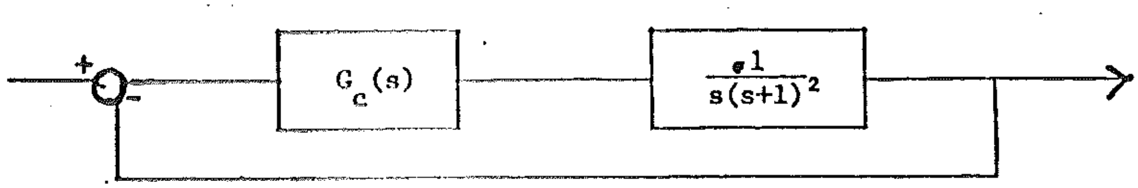
4. *Use (5) to find b/a from $\phi_{max}$ and (7) to find the gain of $(1 + s/a)/(1 + s/b)$ at $\omega = \omega_{max}$.*

5. *Go to the point on the gain-compensated response where the amplitude is down 10 log(b/a). Let this frequency be $\omega_{max}$.*

6. *Use $\omega^2_{max} = ab$ and the already-determined number for b/a to find a and b.*

7. *Compute the phase margin or draw the phase plot of the compensated system to check on the actual phase margin. If it is too low, pick a larger $\phi_{max}$ and go to step 4. If it is too high, pick a lower value of $\phi_{max}$.*

**Example**




**Specifications:**

Phase margin = 45°  Velocity constant = 1 $sec^{-1}$

We now follow the steps outlined in Table III.

1. The straight-line Bode plots are labeled (1) in Figure 3.

2. The velocity constant of the compensated system will be 0.1 K. To meet the error constant spec, we choose K = 10 and draw the new 0 db (labeled (2)) on the Bode plot. The amplitude response relative to this line is called the gain-compensated system.

3. The phase of the gain-compensated system at crossover (1 rad/sec) is 180°. We need 45° lead to have a phase margin of 45°. Add 10% because, as will be seen, the crossover frequency of the final system will be somewhat higher than 1 rad/sec. The 10% is to account for the additional lag from the gain-compensated system that occurs between 1 rad/sec and the as-yet-undetermined final crossover frequency. If we had a phase characteristic that was increasing with frequency (as with nonminimum phase systems) then we would subtract 10% since the gain-compensated system will have less phase lag at the higher crossover frequency.

4. $$\sin\phi_{max} = \frac{\frac{b}{a} - 1}{\frac{b}{a} + 1} = \sin 50° \qquad b/a = 8.0$$

   gain of $(1 + s/a)/(1 + s/b)$ at $\omega_{max} = 10 \log \frac{b}{a} = 9$ db

5. Go to the point where the gain-compensated system is <u>down</u> 10 log(b/a) = 9 db. If we choose this frequency to be $\omega_{max}$, then this will also be the new crossover frequency since the compensator is <u>up</u> 9 db; the sum of the two responses will then be 0 db. The -9 db line is labeled (5) in Figure 2, and we see that $\omega_{max}$ is about 1.6 rad/sec.

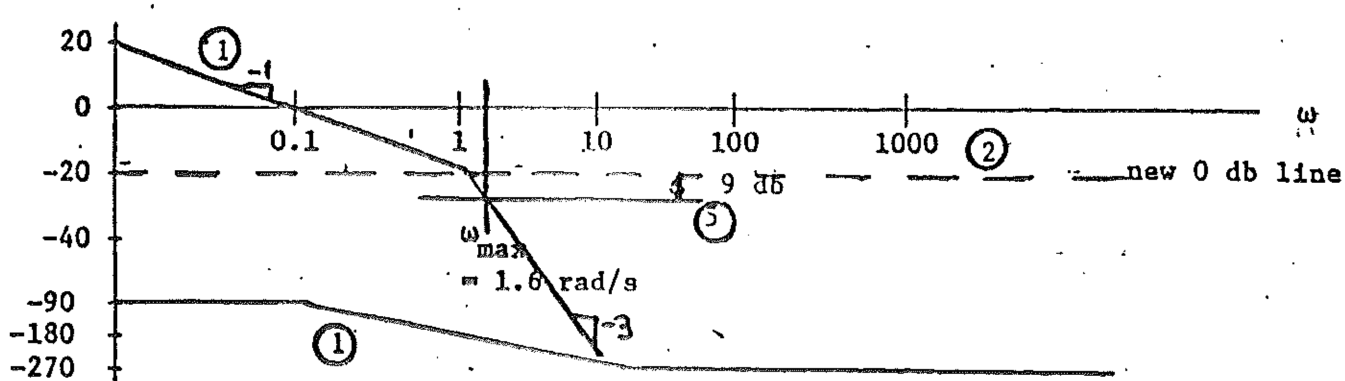



Figure 3. Lead compensation of $0.1/s(s+1)^2$

6. $\omega_{max}^2 = (1.6)^2 = ab$, $b/a = 8$ (from step 4)

   $b^2 = (b/a)(ab) = 8(1.6)^2$ , $b = \sqrt{20.5} = 4.5$

   $a = b(a/b) = 4.5/8 = 0.56$

7. Now we compute the phase of the system at the new crossover frequency, 1.6 rad/sec. (The actual crossover frequency is a little bit lower because we have not made corrections to the straightline Bode plots.)

$$\phi = -90° - 2\tan^{-1}(1.6/1) + \tan^{-1}(1.6/0.56) - \tan^{-1}(1.6/4.5)$$

$$= -90° - 2(58°) + 71° - 21° = -156°$$

phase margin = 180° - 155° = 24°

The phase margin does not meet the specifications because not enough lead is provided by the compensator. From the above expression for the system phase lag we see that an additional 26° of lag was encountered between $\omega = 1.0$ and $\omega = 1.6$. The 10% fudge factor was not enough and this is likely to be the case when the crossover frequency of the gain compensated system is in the vicinity of a pole.

The next guess for $\phi_{max}$ might assume that the crossover of the final design is 1.6 rad/sec so that a total lead of 71° is needed. (This would be implemented by cascading two lead compensators.) In any event the actual crossover will be somewhat to the right of 1.6 rad/sec becaus[e] we are using a larger lead ratio b/a. Therefore, let us use two cascaded lead compensators each giving a maximum lead of 40°, or 80° total. Now we resume the design procedure at step 4 (called 4' in this second iteration).

4'. $\phi_{max} = 40°$ $\quad b/a = 4.5$

gain of $(1 + s/a)^2/(1 + s/b)^2 = 2(10 \log 4.5) = 13.0$ db

5'. Although not shown in Figure 3, we go to the point where the gain-compensated system is down 13 db. This occurs at $\omega = 1.86$ rad/sec, which we choose to be $\omega_{max}$.

6'. $(\omega_{max})^2 = (1.86)^2 = ab$; from 4', $b/a = 4.5$

$b^2 = (ab)(b/a) \quad b = 3.95, \ a = 0.88$

7'. $\phi = -90° - 2\tan^{-1}(1.86/1) + 2\tan^{-1}(1.86/0.88) - 2\tan^{-1}(1.86/3.95)$

$$= -90° - 2(62°) + 2(65°) - 2(25°) = -132°.$$

Phase margin = 180° - 132° = 48°

which is close to the specification of 45°. The compensator is

$$G_c(s) = 10 \frac{(1 + s/0.88)^2}{(1 + s/3.95)^2}$$

## IV. To Meet Error Constant, Bandwidth, and Phase Margin Specs

In the last two situations we found that we could not specify all of the quantities of interest with a lead compensator alone. To give us the additional freedom, we may use a lag-lead network which is just the cascading of lag and lead compensators, although the network need not be realized this way. To meet the error constant, bandwidth, and phase margin specs simultaneously, we use a combination of procedures outlined in Sections I and II as summarized below.

*TABLE IV. TO MEET ERROR CONSTANT, BANDWIDTH, AND PHASE MARGIN SPECS*

1. *Meet the bandwidth and phase margin specs using a lead network as outlined in Section II.*

2. *If the error constant spec is not satisfied by the change in unspecified gain resulting from step 1, then design a lag compensator for the lead-compensated system to give the desired amount of gain increase.*

## DESIGN OF CASCADE COMPENSATORS USING S-PLANE TECHNIQUES

The design of compensators in the s-plane hinges on the placement of the closed-loop roots. Time domain specifications, e.g. percent overshoot, time of first peak (peak time), are translated into an equivalent set of poles; this set is usually a second order complex pair. The compensator is then chosen so that the closed-loop roots of the system are in the vicinity of the desired roots, and these roots should dominate the response of the system

Like the design procedures using Bode plots, we have different "situatio Here, though, the situations are classified into two groups depending on whether or not the compensated root locus passes through the desired closed-loop root location. If the uncompensated root locus does go through the desired closed-loop root location, then the transient response specifications are satisfied. It may be, however, that the error constant specs are not; this is the first procedure to be examined.

### V. To Meet Error Constant Specs But Not Alter Closed Loop Transient Respons

Since the closed-loop transient response is satisfactory, we don't want to influence the path of the root locus in the vicinity of the desired closed-loop roots. This is analogous to Section I where we wanted to meet the error constant spec without changing closed-loop bandwidth or phase margin. Not surprisingly, we use a phase lag network in both situations.

To meet the objectives, the compensator must be chosen so that:

(1) The compensated root-locus passes "near" (to be defined later) the closed-loop root.

(2) When the root locus is near the desired point, the Bode gain, or equivalently, the root locus gain, should be greater than some specified value.

We will look at the gain problem (item (2)) first. Assume that the uncompensated portion of the system has a transfer function

$$GH(s) = K_1 \frac{\prod_i (\frac{s}{z_i} + 1)}{s^n \prod_j (\frac{s}{p_j} + 1)} \tag{8}$$

$$= K_1 \frac{\prod_j p_j}{\prod_i z_i} \frac{\prod (s + z_i)}{s^n \prod (s + p_j)} \tag{9}$$

where $K_1$ is a constant derived from the system equations, and that the compensator has the transfer function

$$G_c(s) = K \frac{(s/a) + 1}{(s/b) + 1} = K(b/a)(\frac{s + a}{s + b}) \tag{10}$$

In the uncompensated system (or more exactly, the gain compensated system), $G_c(s) = K$, and K is chosen so that the closed-loop poles are located as some desired value $s_1$. This value, $K_u$, is determined from the root locus gain condition as follows:

$$1 = K_u K_1 \frac{\prod_j p_j}{\prod_i z_i} \frac{\prod_i |s_1 + z_i|}{|s_1|^n \prod_j |s_1 + p_j|}$$

$$K_u = \frac{1}{K_1} \frac{\prod_i z_i}{\prod_j p_j} \frac{|s_1|^n \prod_j |s_1 + p_j|}{\prod_i |s_1 + z_i|} \tag{11}$$

With the compensator in the loop, and with the assumption that the root locus will pass close to $s_1$, the value of compensator gain to put the closed-loop root at $s_1$, is $K_c$

$$1 \cong K_1 \frac{\prod p_j}{\prod z_i} \frac{\prod_i |s_1 + z_i|}{|s_1|^n \prod |s_1 + p_j|} K_c \frac{b}{a} \frac{|s_1 + a|}{|s_1 + b|}$$

This is an approximation because the root locus cannot pass exactly through $s_1$ with the compensator in the loop. Using (11) we have

$$1 = \frac{K_c}{K_u} \frac{b}{a} \frac{|s_1 + a|}{|s_1 + b|} \tag{12}$$

If we place the pole and zero very close together as shown in Figure 4, then the vectors from the pole and zero are nearly the same length, and (12) becomes

$$\frac{a}{b} \simeq \frac{K_c}{K_u} \tag{13}$$

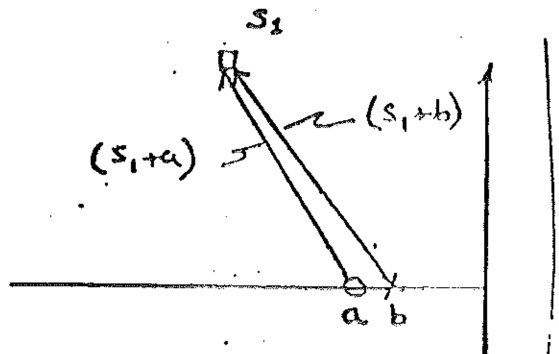



Figure 4. Vectors of Lag Compensator to Desired Closed-Loop Root

With $K_c$ determined from the error constant specification, we now have the pole-zero ratio for the compensator. However, we need two conditions to determine the two parameters a and b. The second condition comes from the requirement that the compensated root locus pass near $s_1$, and this problem is considered next.

Figure 5 shows the geometry of adding a pole-zero compensator: $s_1$ is the desired closed-loop root location; defines the damping constant $\zeta$ ($\zeta = \cos\phi$);

a and b are the zero and pole locations of the compensator; $\theta_1$ and $\theta_2$ are the angles of the vectors from the compensator zero and pole to the closed loop root, respectively; $\Delta\theta = \theta_1 - \theta_2$ is the net contribution of the compensator toward satisfying the root locus angle criterion at the point $s = s_1$. ($\lambda$ is a variable to be used later).

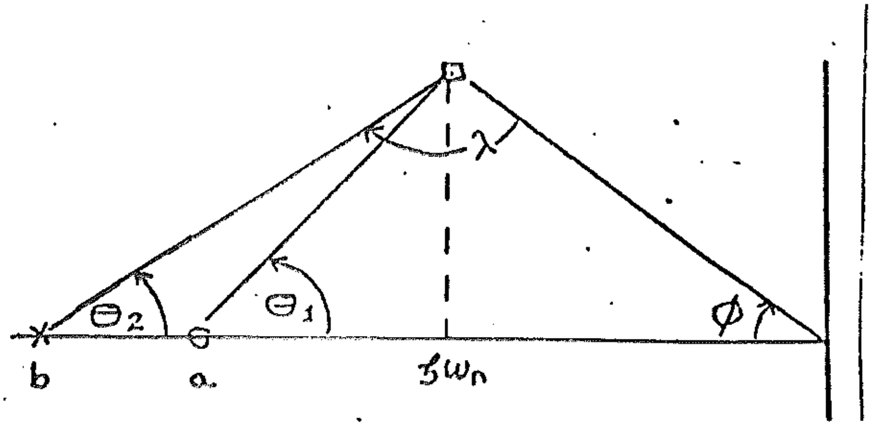



Figure 5. Geometry of a Pole-Zero Compensator

To keep the root locus near $s_1$ we would like to place a and b so that $\Delta\theta$ is small. We may find $\Delta\theta$ as follows. First

$$\tan\theta_1 = \frac{\omega_n\sqrt{1-\zeta^2}}{a-\zeta\omega_n} = \frac{\tan\phi}{\alpha-1} \tag{14}$$

$$\tan\theta_2 = \frac{\omega_n\sqrt{1-\zeta^2}}{b-\zeta\omega_n} = \frac{\tan\phi}{\beta-1} \tag{15}$$

where

$$\tan\phi = \sqrt{1-\zeta^2}/\zeta \qquad \alpha = a/\zeta\omega_n \qquad \beta = b/\zeta\omega_n \tag{16}$$

Using the trigonometric relations for the tangent of the sum of two angles yields:

$$\tan(\theta_1-\theta_2) = \tan\Delta\theta = \frac{\tan\phi\left(\frac{1}{\alpha-1}-\frac{1}{\beta-1}\right)}{1-\tan^2\phi\left(\frac{1}{\alpha-1}\right)\left(\frac{1}{\beta-1}\right)}$$

$$= \frac{\sin\phi(\beta-\alpha)}{\cos\phi(\alpha\beta-\alpha-\beta+1)+\frac{1}{\cos\phi}}$$

$$\tan\Delta\theta = \frac{\sin\phi\left(\frac{\beta}{\alpha}-1\right)}{\cos\phi\left(\frac{\beta}{\alpha}(\alpha-1)-1\right)+\frac{1}{\alpha\cos\phi}} \tag{17}$$

Now we want to choose $\alpha$ so that $\Delta\theta$ will be less than some prescribed value. To do this, we make $\alpha$ very small and with $\beta/2$ constant (17) becomes

$$\tan\Delta\theta \simeq \left(\frac{b}{a}-1\right)\alpha\sin\phi\cos\phi \tag{18}$$

where we have used the fact that $\alpha/\beta = b/a$. To have $\Delta\theta$ less than some criterion (say 2° or 5°), we then want $|\tan\Delta\theta| < \varepsilon$; so

$$\alpha|(\frac{b}{a} - 1)|\sin\phi\cos\phi < \varepsilon \qquad \alpha = \frac{a}{\zeta\omega_n} < \varepsilon/|\frac{b}{a} - 1|\sin\phi\cos\phi \tag{19}$$

or

$$a/\omega_n < \varepsilon/|\frac{b}{a} - 1|\sqrt{1 - \zeta^2} \tag{20}$$

The summary of this design procedure is contained in Table V.

*TABLE V. TO MEET AN ERROR CONSTANT SPEC BUT NOT ALTER CLOSED-LOOP TRANSIENT RESPONSE (LAG NETWORK)*

1. *Set the compensator gain K to obtain the required increase in error constant.*
2. *Choose the ratio a/b from a/b = K.*
3. *Choose the value of a such that*

$$a/\omega_n < \frac{|\tan\Delta\theta_{max}|}{|\frac{b}{a} - 1|\sqrt{1 - \zeta^2}}$$

*where ($\zeta$, $\omega_n$) describe the location of the desired closed-loop roots, and $\Delta\theta_{max}$ is the maximum allowable deviation in the root locus angle criterion at the closed-loop root.*

## VI. To Alter the Root Locus So That It Passes Through A Specified Point

If the time domain specifications are translated to a pair of complex roots that do not lie on the uncompensated root locus, then a compensator (usually a lead compensator) is required. Again denote the desired closed-loop root by $s_1$; the uncompensated transfer function GH(s); and the compensator as $G_c(s)$. to have the compensated root locus go through $s_1$, the (180°) angle condition for the root locus requires that

$$180° = \measuredangle GH(s_1) + \measuredangle G_c(s_1) \quad \text{or} \quad \measuredangle G_c(s_1) = 180° - \measuredangle GH(s_1) \tag{21}$$

This deficiency in the angle criterion can be easily computed and must be overcome by the compensation network. Refer back to Figure 5 and let the quantity $\Delta\theta$ be equal to

$$\Delta\theta = 180° - \measuredangle GH(s_1) \tag{22}$$

Then any placement of the compensator pole and zero using this value of $\Delta\theta$ will cause the root locus to go through the point $s_1$. However, one must be sure that the closed-loop roots dominate the response. As a rule of thumb, the zero should be to the left of the first real pole in a Type 1 system, and to the left of the second real pole in a Type 0 system. Sketch a root locus for these cases and see why the rule of thumb is used. Another factor to consider is that the compensator zero may contribute more overshoot than is desired. There is a curve in Clark (p. 123) which presents percent overshoot for a second order system with a zero. This may be used as a guide to indicate whether or not the overshoot of the compensated system will be excessive. Before proceeding to specific design procedures, let us examine some characteristics of the lead compensator.

### Gain Required to Obtain Desired Closed-Loop Roots

The gain $K_c$ required to make the root locus pass through $s_1$ is given in (12) and is repeated here.

$$\frac{K_c}{K_u} = \frac{a}{b}\frac{B}{A} \tag{23}$$

where $K_u$ is a constant computed from (1) and (see Figure 5).

$$A = |s_1 + a| \qquad B = |s_1 + b|$$

The following trigonometric identities come from Figure 5:

$$\frac{\sin\phi}{B} = \frac{\sin\lambda}{b} \qquad \frac{\sin\phi}{A} = \frac{\sin(\lambda - \Delta\theta)}{a} \tag{24}$$

Substituting them in (23) yields

$$\frac{K_c}{K_u} = \cos\Delta\theta - \sin\Delta\theta\cot\lambda = \sin\Delta\theta(\cot\Delta\theta - \cot\lambda) \qquad \Delta\theta \leq \lambda \leq \Pi - \phi \tag{25}$$

This shows that the gain required is a monotonic function of the zero location. The zero should be placed as far as possible to the left to make $K_c$, and thus the error constant, as large as possible.

### Lead Ratio b/a

Another parameter of interest is the lead ratio, b/a. This should generally be limited to (say) 10 or 15 because of implementation problems, although higher values are permissible in many instances. To compute the lead ratio we use the identities (see Figure 5):

$$\sin\lambda/b = \sin\theta_2/\omega_n \qquad \sin(\lambda - \Delta\theta)/a = \sin\theta_1/\omega_n \tag{26}$$

$$\theta_2 = \Pi - (\phi + \lambda) \qquad \theta_1 = \Pi - (\phi + \lambda - \Delta\theta) \tag{27}$$

Equations (26) and (27) can be combined to yield

$$a/\omega_n = \sin(\lambda - \Delta\theta)/\sin(\lambda - \Delta\theta + \phi) \qquad b/\omega_n = \sin\lambda/\sin(\lambda + \phi) \tag{28}$$

The lead ratio b/a is thus

$$\frac{b}{a} = \frac{\sin\theta_1}{\sin(\lambda - \Delta\theta)} \cdot \frac{\sin\lambda}{\sin\theta_2} = \frac{\cos\phi + \sin\phi\cot(\lambda - \Delta\theta)}{\cos\phi + \sin\phi\cot\lambda} \tag{29}$$

This becomes infinite at two points: $\lambda = \Delta\theta$, or $a = 0$; and $\theta_2 = 0$, or $b = \infty$. There is a minimum at the point $\lambda = (\Pi/2) - (\phi/2) - (\Delta\theta/2)$ such that the angles to the pole and zero are placed at equal angles about the bisector of the angle defined by a line from $s_1$ to the origin of the s-plane and a line from $s_1$ toward $-\infty$ which is parallel to the negative real axis (D'Azzo and Houpis, p. 412).

Since there is one unspecified parameter in the choice of lead network, we will describe several design requirements and the procedures to meet them which are typical of ones encountered in practice.

1. <u>To bring the root locus to a specified point with a minimum lead ratio, b/a.</u>

The lead ratio is sometimes minimized because the network implementation requires an amplifier gain which is b/a times the compensator gain. In other words, the lead network has a d.c. gain of a/b, so that minimizing the lead ratio minimizes the attenuation at d.c. The design procedure for this case is summarized in Table VI.

*TABLE VI. TO BRING THE ROOT LOCUS TO A SPECIFIED POINT WITH A MINIMUM LEAD RATIO, b/a.*

| | |
|---|---|
| 1. | *Compute* $\phi$, $\Delta\theta$ *and let* $\lambda = \frac{1}{2}(\Pi - \phi + \Delta\theta)$ *to give minimum b/a.* |
| 2. | *Solve (28) for b and a.* |
| 3. | *Find the compensator gain* $K_c$ *from (12).* |
| 4. | *Check the design for stability and specifications. If the specifications are not met, then choose new points for the dominant closed-loop roots and go to Step 1.* |

It may turn out that this minimum lead ratio is too large from a practical standpoint, and two or more lead compensators are needed. If m compensators are used, one can replace $\Delta\theta$ by $\Delta\theta/m$ in the above procedure. The gain computation, too, must be altered.

2. <u>To bring the root locus to a specified point, and maximize the error constant for some maximum lead ratio.</u>

In this case, we are supposing that we would like to get the maximum increase in the error constant, but we do not want to exceed a certain lead ratio because of hardware considerations. We increase the error constant by increasing the Bode gain, and as we have previously seen, this implies that we should choose the compensator such that the zero is as far to the left as possible. The maximum allowable gain occurs when the lead ratio is at its maximum value. The details of this procedure are given in Table VI.

*TABLE VI. TO BRING THE ROOT LOCUS TO A DESIRED POINT AND MAXIMIZE THE ERROR CONSTANT FOR SOME MAXIMUM LEAD RATIO*

1. *Compute* $\phi$ *and* $\Delta\theta$.
2. *Choose the larger value of* $\lambda$ *which satisfies (29) with b/a at its maximum value.*
3. *Solve (28) for a and b.*
4. *Find the compensator gain* $K_c$ *from (12).*
5. *Check the designs for stability and specifications. If the specifications are not met, then choose new points for the dominant closed-loop roots and go to step 1.*

3. <u>To bring the root locus to a specified point with a given error constant</u>

This procedure can be used frequently since, if it can be done, it simultaneously satisfies the transient response and error constant specification

*TABLE VII. TO BRING THE ROOT LOCUS TO A SPECIFIED POINT WITH A GIVEN ERROR CONSTANT*

1. *Compute* $\phi$ *and* $\Delta\theta$.
2. *Compute the required compensator gain* $K_c$ *from the error constant requirement.*
3. *Find* $\lambda$ *from (25) and a and b from (28).*
4. *If the lead ratio is not too large, and* $\lambda$ *is within the allowable range, then check the design for stability and specifications and if they are not met, modify the positions of the dominant closed-loop roots and go to Step 1.*
5. *If the lead ratio is too large for one compensator, two alternatives are possible: cascading two or more lead compensators, or using a lag-lead compensator. If the latter choice is made, the lead portion may be chosen by either of the criteria discussed previously (Table VI or VII) and the lag portion can be chosen to make up the deficiency in error constant (see Table V). Regardless of the course of action, the final design should be checked for stability and specifications.*

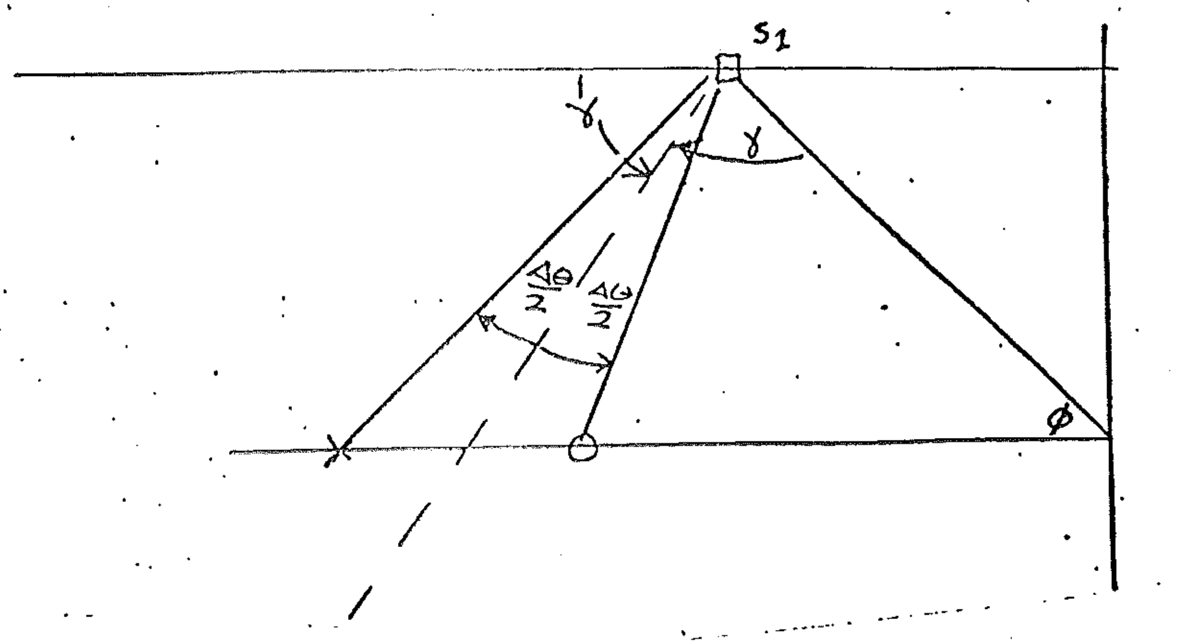



Figure 6. Pole-zero placement to minimize lead ratio b/a.

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