CMPE-242 Practice Final Exam
02-Dec-2010

Problem 1: Internal stability (10 points).

A system's input output transfer function is

$$G(s) = \frac{1}{s^2(s+2)^3(s+1)}$$

Is the system internally stable?

Problem 2: Controllability and Observability (10 points).

(a) Suppose that you are given the controllable system:

$$\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx \tag{2}$$

and apply a similarity transform to this system so that it is represented with respect to new state $\overline{x} = Px$. Prove that the new system representation is controllable.

(b) A state space representation of a system in controllable canonical form is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{3}$$
$$y = [0.8\ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4}$$

Is this system, as represented in controllable canonical form, observable?

(c) The same system may be represented by the following state space equations, which is in observable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1.3 & 1 \\ -0.4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} u \tag{5}$$
$$y = [1\ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6}$$

In this form, is the system controllable?

(d) Explain what causes the apparent differences in both controllability and observability of the same system.

Problem 3: State vs. output feedback (8 points).

The linearized and normalized longitudinal motion of a helicopter near hover can be modeled as a normalized third order system:

$$\begin{bmatrix} \dot{q} \\ \dot{\theta} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 0 \\ 2 & 10 & -1 \end{bmatrix} \begin{bmatrix} q \\ \theta \\ v \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 10 \end{bmatrix} \delta \tag{7}$$

where q is the pitch rate, θ is the pitch, v is the horizontal aircraft velocity, and δ is the rotor tilt angle, δ is considered the input to this system.

(a) Is the system controllable?

(b) Assuming that all state variables are available for measurement, compute the state feedback gain matrix $F = [f_1\ f_2\ f_3]$ to place the closed loop eigenvalues at:

$$-2, -1 \pm 1j$$

(c) Now, suppose instead that only the output $y = v$ is available for measurement. Consider a classical control design which uses output feedback $u(t) = r - ky(t)$ where k is a scalar, proportional gain. Is there a value of k for which the response of the closed loop system is identical to that of the design of part (b)?

Problem 4: Observer design I (10 points).

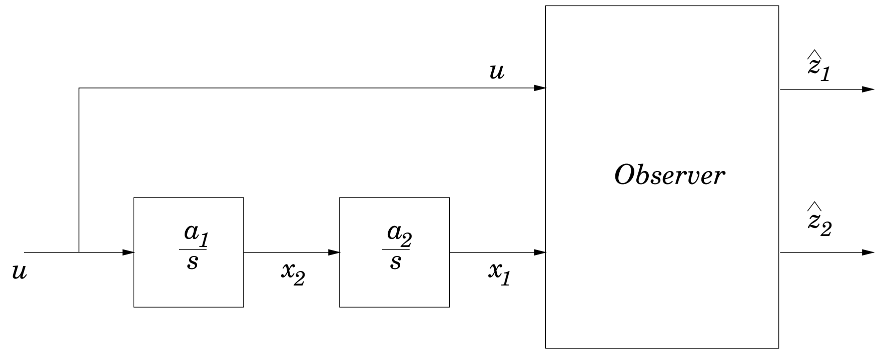

Figure 1: Simple model of a DC Servo system, for Problem 4.

Figure 1 shows a block diagram representation of a simple model of a DC servo system: x_1 is a voltage signal proportional to the output angular velocity x_2.

(a) Design a full order observer, with observer gain matrix T given by

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \tag{8}$$

for x_1 and x_2 so that the characteristic polynomial associated with the error dynamics is given by:

$$\Delta_e(s) = s^2 + 2\zeta_e\omega_e s + \omega_e^2 \tag{9}$$

("Design" means write the equations for the observer, with expressions for gains T_1 and T_2.)

(b) Now, the observer is a system with inputs u and x_1, and outputs $\hat{z_1}$ and $\hat{z_2}$. Thus, there are four possible transfer functions between inputs and outputs – these may be included as elements in a 2×2 matrix. Evaluate the following *matrix of transfer functions* $M(s)$ between the inputs to the observer u and x_1, and its outputs $\hat{z_1}$ and $\hat{z_2}$:

$$M(s) = \begin{bmatrix} \hat{z}_1(s)/u(s) & \hat{z}_1(s)/x_1(s) \\ \hat{z}_2(s)/u(s) & \hat{z}_2(s)/x_1(s) \end{bmatrix} \tag{10}$$

as a function of gains T_1 and T_2, as well as system parameters a_1 and a_2.

(c) Now determine $M(s)$ as $T_2 \to \infty$. Discuss the meaning of the result.

Problem 5: Observer design II (10 points).

Design an observer for the following *nonlinear* time invariant system:

$$\begin{aligned} \dot{X} &= AX + BU + f_1(CX) \\ Y &= f_2(CX) \end{aligned}$$

where $A \in \Re^{n\times n}, B \in \Re^{n\times n_i}, C \in \Re^{n_o\times n}$, (A, C) is observable, $f_1 : \Re^{n_o} \rightarrow \Re^{n}$ is a *known* nonlinear function, and $f_2 : \Re^{n_o} \rightarrow \Re^{n_o}$ is a *known, invertible* nonlinear function (meaning that given Y, you can determine CX as $f_2^{-1}(Y)$).

It is sufficient to show the block diagram, as well as the state space equations, of the observer.

Problem 6: Observer design III (10 points).

You are given a system

$$\begin{aligned} \dot{X} &= AX + BU \\ Y &= CX \end{aligned}$$

which is both controllable and observable. You design a full order observer with gain matrix T such that $(A - TC)$ is stable; denoting observer state $\hat{Z}$, you design a feedback gain matrix F, such that $U = R - F\hat{Z}$, where R is the reference input and $(A - BF)$ is stable. Denote the state estimate error as $E = \hat{Z} - X$.

(a) Derive the transfer function from the reference input R to the output Y, in terms of the plant and gain matrices. What is striking about this transfer function?

(b) Derive the transfer function from the reference input R to the error E. Comment about this transfer function.

(c) Suppose that the plant changes by a small amount δA. What is the resulting $H_{EU}(s)$? Does the error E still go to zero as $t \to \infty$?

Problem 7: How does state feedback affect observability? (10 points)

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{11}$$

$$y = [1 \;\; 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{12}$$

and assume that you would like to design a feedback controller of the form $u = -Fx + r$, where $x = [x_1 \; x_2]^T$ and r is a reference input signal.

(a) Show that the system is observable.

(b) Show that there exists a state feedback gain matrix $F = [f_1 \; f_2]$ such that the closed loop system resulting from setting $u = -Fx + r$ is not observable.

(c) Now, compute a matrix F of the form $F = [1 \; f_2]$ such that the closed loop system (as in part (b)) is not observable.

(d) By comparing the open loop transfer function with the transfer function of the closed loop system of part (c), state what this unobservability is due to.

Problem 9: Design I (10 points).

Consider the system shown in Figure 2.

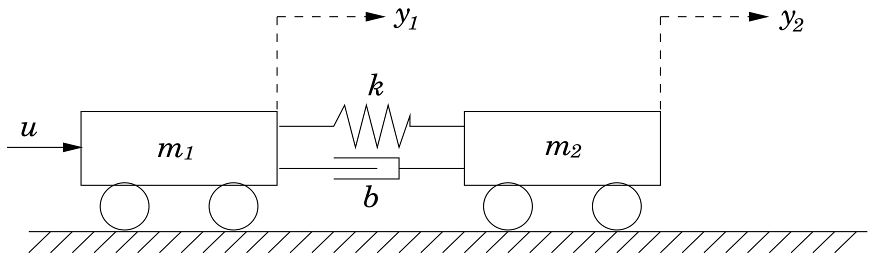

Figure 2: Two cart mechanical system, for Problem 9.

Assume numerical values of masses $m_1 = 1kg$, $m_2 = 2kg$, spring constant $k = 36N/m$, and damping coefficient $b = 0.6Ns/m$.

Only the positions of the two carts (y_1 and y_2) are available for measurement; the input u is an external force that is applied as shown to m_1. Using a full state observer, and controller designed using pole placement, design a regulator system so that the system will maintain zero position ($y_1 = 0$ and $y_2 = 0$) in the presence of disturbances. Choose the desired closed loop poles for the system to be at:

$$\{-2 + j2\sqrt{3}, -2 - j2\sqrt{3}, -10, -10\}$$

Justify the observer eigenvalue locations. Plot the initial state response for both the open loop and closed loop system, for a non-zero initial position of y_1 and y_2.

Problem 10: Design II (12 points).

Consider the system shown in Figure 3.

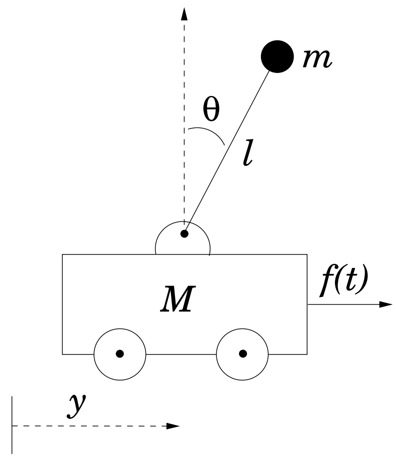

Figure 3: Inverted pendulum on a cart, for Problem 10.

Assume numerical values of $M = 2kg$, $m = 0.1kg$, $l = 0.5m$.

Here, we are concerned with the motion of the pendulum and the cart, in the plane of this page.

It is desired to keep the inverted pendulum upright as much as possible, by controlling the position of the cart. To control the position of the cart, we measure the cart position y, and we feed it back to the input and insert an integrator in the feedforward path, as shown in Figure 4.

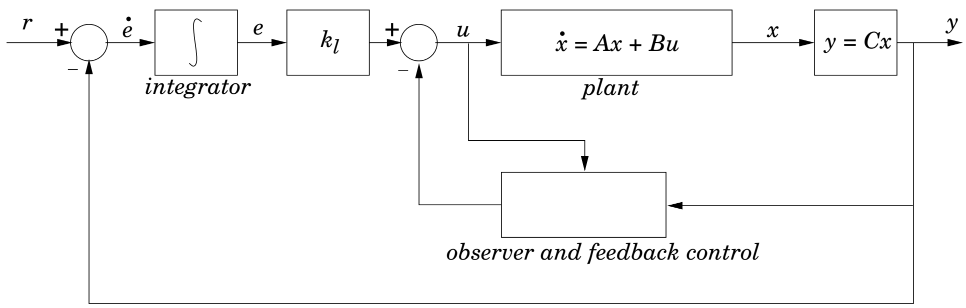

Figure 4: Inverted pendulum control system, with integral control, for Problem 10.

We may assume that the pendulum angle θ and angular velocity $\dot{\theta}$ are small, so that the equations for the inverted pendulum control system are:

$$M l \ddot{\theta} = (M+m) g \theta - u \tag{13}$$

$$M \ddot{y} = u - m g \theta \tag{14}$$

(a) Using as state variables $x_1 = \theta, x_2 = \dot{\theta}, x_3 = y, x_4 = \dot{y}$, and $x_5 = e$, and letting the single measured output y be the cart position ($y = x_3$), obtain the state space representation of the system shown in Figure 4, when the box marked "observer and feedback control" is empty. Explain the reason for including integral control here.

(b) Now, design an "observer and feedback control" system, such that the closed loop poles are at:

$$\{-1 + j\sqrt{3}, -1 - j\sqrt{3}, -5, -5, -5\}$$

(which results in a settling time for θ of approximately $4 - 5$ seconds, with an overshoot of not more than about 15%).

Include the equations and gain matrices of your "observer and feedback control" system in your response. Justify the choice of your observer poles.

(c) Plot the response of the system to a step input $r(t)$.