CMPE 240 Office Hours

Last ones ☺

① $Ax = b$ A full rank $x \in \mathbb{R}^{1000000}$ BIG

A_3 can be computed easily, but $x = A^{-1}b$ not so.

$\hat{A}^{-1}$ is easy A^{-1} is hard

$$\hat{x}(0) = \hat{A}^{-1} b$$

$$r = A\hat{x}(j) - b$$

$$\hat{x}(j+1) = \hat{x}(j) - \hat{A}^{-1} r(j)$$

$\Theta \quad \beta = \| \hat{A}^{-1}(A - \hat{A}) \| \quad \text{if } \hat{x}(0) = \hat{A}^{-1} b$

$\| \hat{x} - x \| \leq \beta^{t+1} \| x \| \quad \text{if } \beta < 1 \text{ then } \hat{x}(t) \to x$

Relaxation

$$\hat{x}_0 = \hat{A}^{-1} b \qquad r_k = A\hat{x}_k - b$$

$$\hat{x}_{k+1} = \hat{x}_k - \hat{A}^{-1} r_k$$

$$\hat{x}_{k+1} = \hat{x}_k - \hat{A}^{-1}[A\hat{x}_k - b]$$

$$\hat{x}_{k+1} = \hat{x}_k - \hat{A}^{-1}A\hat{x}_k + \hat{A}^{-1}b$$

$$= \underbrace{(I - \hat{A}^{-1}A)}_{L}\hat{x}_k + \underbrace{\hat{A}^{-1}b}_{\hat{x}_0}$$

eig ⊕

Form $\tilde{x}_k = \hat{x}_k - x$

but $x = A^{-1}b$

$$\tilde{x}_{k+1} = \hat{x}_{k+1} - A^{-1}b$$

$$\tilde{x}_{k+1} = (I - \hat{A}^{-1}A)\hat{x}_k + \hat{A}^{-1}b$$

↑ $-A^{-1}b$

$$\tilde{x}_{k+1} = [I - \hat{A}^{-1}A](\tilde{x}_k + x) + (\hat{A}^{-1} - A^{-1})b$$

$$\tilde{x}_k = \hat{x}_k - x$$

$$\hat{x}_k = \tilde{x}_k + x$$

$$\tilde{x}_{k+1} = [I - \hat{A}^{-1}A]\tilde{x}_k + Ix - \hat{A}^{-1}\underbrace{Ax}_{b} + \hat{A}^{-1}b - A^{-1}b$$

$$\hat{A}^{-1}b$$

$$\tilde{x}_{k+1} = [I - \hat{A}^{-1}A]\tilde{x}_k$$

$$\boxed{\tilde{x}_{k+1} = \underbrace{\hat{A}^{-1}[\hat{A} - A]}_{\phi}\tilde{x}_k}$$

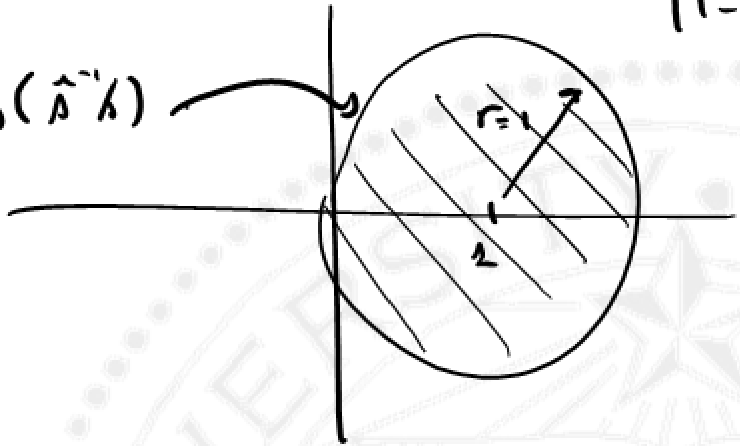
$$\tilde{x}_{k+1} = \phi\,\tilde{x}_k.$$

$$|\mathrm{eig}(\phi)| < 1.$$

(b) $\text{eig}(I - \hat{A}^{-1}A)$ have form $1 - \lambda_i$ $\quad \lambda_i : \text{eig}(\hat{A}^{-1}A)$.

$$|1 - \lambda_i| < 1$$

$\lambda : \text{eig}(\hat{A}^{-1}A)$



(c)

$$\tilde{x}_{k+1} = \hat{A}^{-1}[\hat{A} - A]\tilde{x}_k$$

$$\|\tilde{x}_{k+1}\| = \|\hat{A}^{-1}(\hat{A} - A)\tilde{x}_k\| \leq \underbrace{\|\hat{A}^{-1}[\hat{A} - A]\|}_{\rho} \cdot \|\tilde{x}_k\|$$

$$\|\tilde{x}_{k+1}\| \leq \rho \|\tilde{x}_k\|$$

$$\|\tilde{x}_k\| \leq \rho^k \|x - \hat{A}^{-1} b\|$$

$$x - \hat{A}^{-1} b = x - \hat{A}^{-1} A x = (I - \hat{A}^{-1} A) x = \underline{\hat{A}^{-1} (\hat{A} - A) x}$$

(5) $y_1 \in \mathbb{R}^n$ $x_1 \in \mathbb{R}^n$

$\vdots$ $\vdots$

y_N x_N

$y_i = x_i + d_i$ $q^T x_i = 0$

$q^T y_i = q^T d_i$

$$\rho = \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N}(q^T y_i)^2}}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\|y_i\|^2}} \leftarrow$$

$$\sqrt{q^T Y Y^T q} = \|Y^T q\|$$

$$Y = [y_1\ y_2 \dots y_N] \in \mathbb{R}^{n \times N}$$

$$\sqrt{\mathrm{Tr}(Y^T Y)}$$

$$\begin{bmatrix} y_1^T \\ \vdots \\ y_N^T \end{bmatrix} [y_1 \dots y_N]$$

$$\rho = \frac{\|Y^T q\|}{\sqrt{\mathrm{Tr}(Y^T Y)}}$$

$Y = [y_1 \cdots y_N] \in \mathbb{R}^{n \times N}$

$N > n$

$N < n$

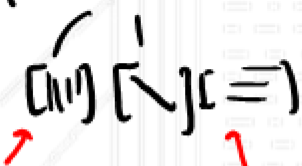
Frobenius norm Y.

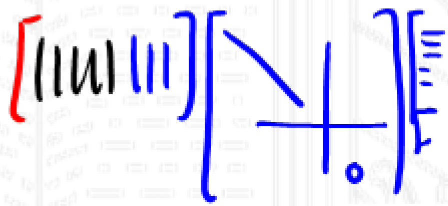
thin SVD: $Y = U \Sigma V^T$ ($N \times N$)

$n \times n$ $\quad \mathbb{R}^{n \times N}$

$$\rho = \frac{\|V \Sigma^T U^T q\|}{\sqrt{\mathrm{Tr}(V \Sigma^2 V^T)}} = \frac{\|\Sigma^T U^T q\|}{}$$

SVD: $Y = U \Sigma V^T$





$$\rho = \frac{\|V\Sigma U^T q\|}{\sqrt{\operatorname{Tr} V\Sigma^2 V^T}} = \frac{\|\Sigma U^T q\|}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \geq \frac{\sigma_n}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}}$$

$$\Sigma U^T q = \sigma_n$$

$q \downarrow u_n$

$q = u_n$ singular vector corresponding to smallest singular value σ_n.

7

$$x_{k+1} = Ax_k + Bu_k \qquad x_0 = x_{\sigma}.$$

$$E(x_0) = \min\left\{\sum_{\tau=0}^{t-1} \|u(\tau)\|^2 \; x(t) = 0\right\}$$

$$z_k = A^{-1} z_{k+1} - A^{-1} B v_k \qquad x_{k+1} = Ax_k + Bu_k$$

$$\mathcal{E}_{z_k} = x_0^T \left[\sum_{\tau=0}^{t_f-1} A^{-(\tau+1)} B B^T A^{-(\tau+1)}\right]^{-1} x_0 = x_0^T \left[\sum_{\tau=1}^{t_f} A^{-\tau} B B^T (A^{-\tau})^T\right] x_0$$

$$\mathcal{E}_x = x_0^T (A^{t_f})^T \left[\sum_{\tau=0}^{t_f-1} A^{\tau} B B^T (A^{\tau})^T\right]^{-1} A^{t_f} x_0$$

$$t = t_f - \tau \longrightarrow \mathcal{E}_z = \mathcal{E}_x$$

(7)

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Controllability:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

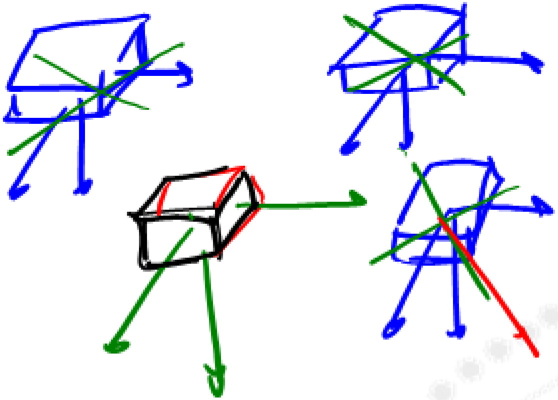
$$x_{k+1} = Ax_k + Bu_k \qquad x_0 = 0$$

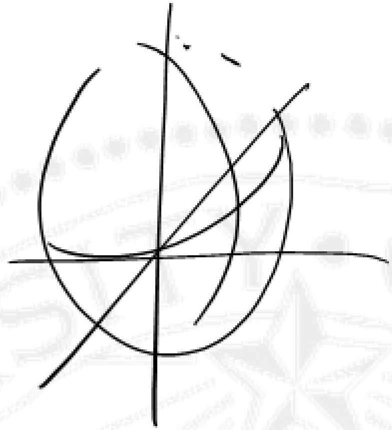
$$x_1 = Ax_0 + Bu_0$$

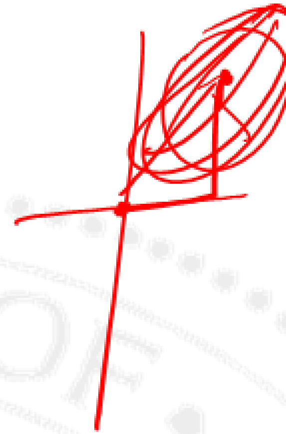
$$x_2 = Ax_1 + Bu_1 = ABu_0 + Bu_1$$

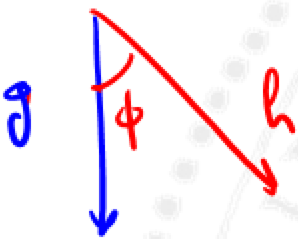
$$x_3 = Ax_2 + Bu_2 = A^2Bu_0 + ABu_1 + Bu_2$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$








g
φ
h