Least-Norm Solutions of Underdetermined Equations

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Least-Norm Solutions of Underdetermined Equations

- Least-norm solution of underdetermined equations
- Minimum norm solutions via QR factorization
- Derivation via Lagrange multipliers
- Relation to regularized least-squares
- General norm minimization for equality constraints

Underdetermined Linear Eq'ns (1.2)

$y = Ax$ each equation is a spec.

$A \in \mathbb{R}^{m \times n}$ is FAT $(m < n)$

- There are fewer equations than unknowns
- x is underspecified, many choices of x give the same y.

Assume A is full rank (m)

For each $y \in \mathbb{R}^m$ there is an exact solution..

Underdetermined Linear Eq'ns (2.2)

$\{x \mid Ax = y\} = \{x_p + z \mid z \in N(A)\}$

x_p = "particular" solution $\underline{Ax_p = y}$.

In design — great news!

z characterizes available choices within solution

Solution has $\dim N(A) = n - m$ "degrees of freedom"

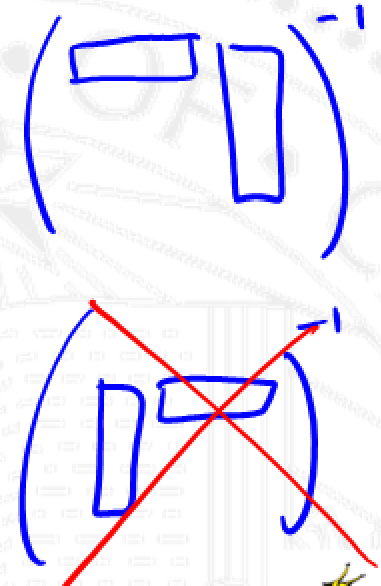
Choose z to satisfy some <u>other</u> specification

or do optimize among many solutions.

Least-norm Solution (1.3)

$x_{ln} = \underbrace{A^T(AA^T)^{-1}}_{\text{right inverse of } A}\, y$ — least norm solution

$x_{ls} = (A^TA)^{-1}A^Ty$



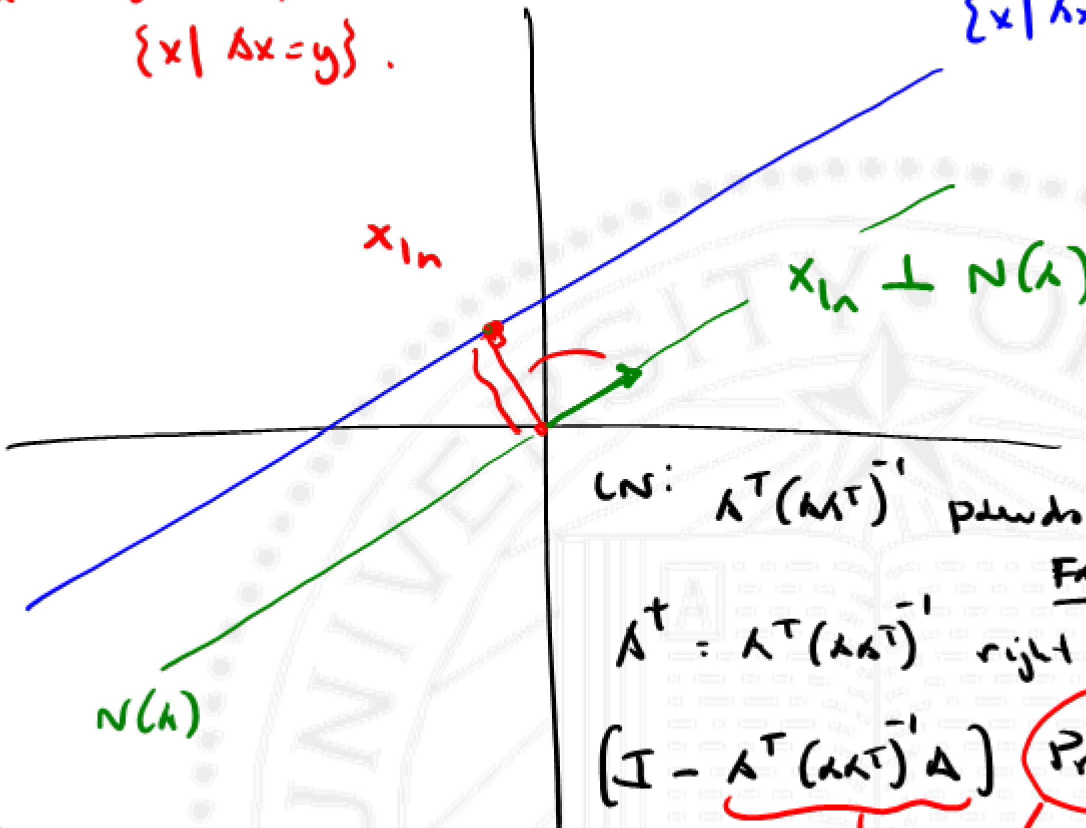
(AA^T) invertible because A full rank

LS: $y \simeq Ax$ (approx)

LN: $y = Ax$ (exactly)

x_{ln} solves $y = Ax$ with $\|x\|^2$ min.

x_{ln} projection of $\{0\}$ onto solution set
$\{x \mid Ax = y\}$.



LN: $A^T(AA^T)^{-1}$ pseudo inverse of FAT A

$A^{\dagger} = A^T(AA^T)^{-1}$ right inv of A

$\left[I - A^T(AA^T)^{-1}A\right] \quad P_r(N(A))$.

Least-norm Solution (2.3)

$$\text{LN} \begin{cases} \min\limits_x \|x\|^2 \\ \text{s.t. } Ax = y \end{cases}$$

$$\text{LS: } \min_x \|Ax - y\|^2$$

$$Ax = y \qquad A(x - x_{ln}) = 0$$

$$(x - x_{ln})^T x_{ln} = \boxed{(x - x_{ln})^T A^T} (AA^T)^{-1} y$$

$$\underbrace{(A(x - x_{ln}))^T}_{=0} (AA^T)^{-1} y$$

$$(x - x_{ln}) \perp x_{ln}$$

Least-norm Solution (3.3)

$$\|x\|^2 = \|x_{ln} + x - x_{ln}\|^2 = \|x_{ln}\|^2 + 2x_{ln}^T \cancelto{0}{(x - x_{ln})} + \|x - x_{ln}\|^2$$

$$\geq \|x_{ln}\|^2$$

$$\|x_{ln}\|^2 + \|x - x_{ln}\|^2 \geq \|x_{ln}\|^2$$

x_{ln} has the smallest norm of any solution.

$y = Ax$

Least-norm Solution via QR (1.3)

find QR of $A^T \longrightarrow A^T = QR$ (G.S on A^T)

$A^T = QR \qquad Q \in \mathbb{R}^{n \times m} \qquad Q^TQ = I_m$

$R \in \mathbb{R}^{m \times m}$ upper triangular, invertible

$$x_{ln} = A^T(AA^T)^{-1}y = QR(R^TQ^TQR)^{-1}y = QR(R^TR)^{-1}y$$

$$= Q\underbrace{RR^{-1}}(R^T)^{-1}y = QR^{-T}y$$

$$R^{-T} \triangleq (R^{-1})^T$$

$$\|x_{ln}\| = \|R^{-T}y\|$$

Derivation via Lagrange Multipliers

$\min x^T x$

subj $Ax = y$

$$L(x,\lambda) = x^T x + \lambda^T (Ax - y)$$

optimality conditions : $\frac{\partial L}{\partial x} = 0 \quad \frac{\partial L}{\partial \lambda} = 0$

$$\frac{\partial L}{\partial x} = 2x + A^T\lambda = 0 \quad \therefore \quad x = -\frac{A^T\lambda}{2}$$

$$\frac{\partial L}{\partial \lambda} = Ax - y = 0$$

Derivation via Lagrange Multipliers

$$Ax - y = 0 \qquad x = -\frac{A^T\lambda}{2}$$

$$-\frac{AA^T\lambda}{2} - y = 0 \longrightarrow \lambda = -2(AA^T)^{-1}y$$

$$x = -\frac{A^T\lambda}{2} = +\frac{A^T}{\cancel{2}}(-\cancel{2}(AA^T)^{-1}y)$$

$$\boxed{x_{ln} = A^T(AA^T)^{-1}y}$$

if A is FAT full rank ... plenty of solutions

- why do we choose $\min \|x\|^2$

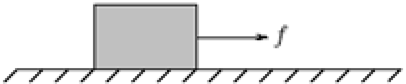
EASY TO CALCULATE

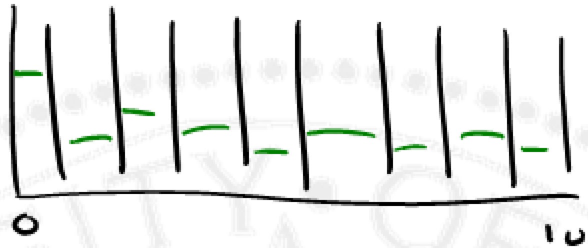
$\min \sum |x_i|$ – 1 norm ~ fuel burn

$\min \max |x_i|$ – ∞ norm ~ thruster size.

H_2 control (LQR/LQG/LQE)

Lagrange Multiplier: Example (1.3)





y_1 position @ $t=10$

y_2 velocity @ $t=10$

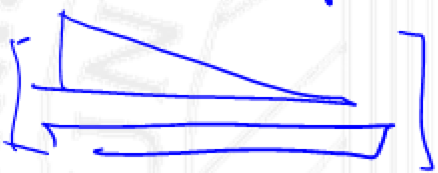
$A = F \times T$

$y = Ax \quad x \in \mathbb{R}^{2\times10}$

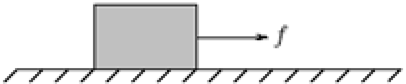
$x \in \mathbb{R}^{10}$

find least norm transfers moves to $y=1$ $\dot{y}=0$

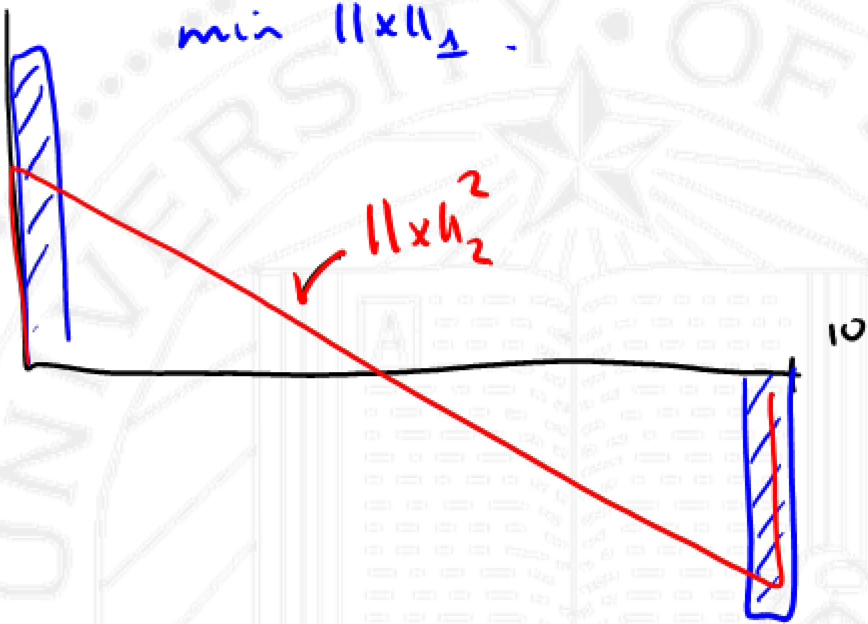
$y_{des} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



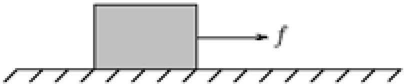
Lagrange Multiplier: Example (2.3)

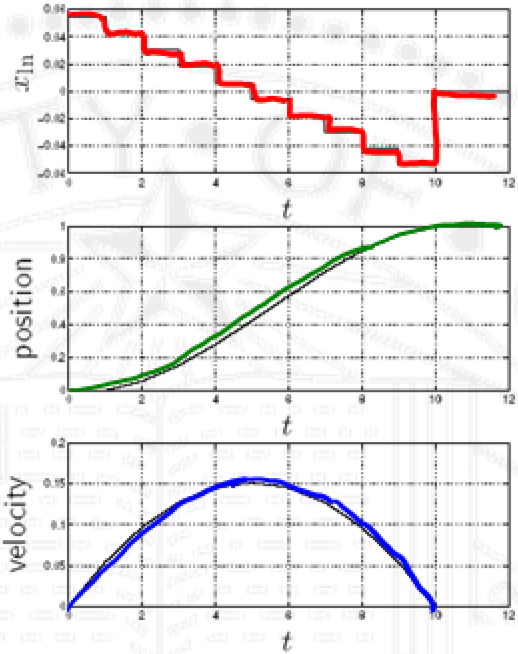


$$x_{ln} = A^T(AA^T)^{-1}\begin{bmatrix}1\\0\end{bmatrix}$$



Lagrange Multiplier: Example (3.3)

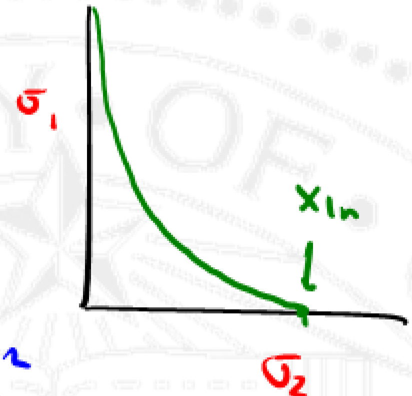




Relation to Regularized L-S (1.2)

$A \in \mathbb{R}^{m \times n}$ $\underline{AA^T}$ full rank

define $J_1 = \|Ax - y\|^2$ $J_2 = \|x\|^2$



<u>Least norm</u> $\min J_2$ w/ $J_1 = 0$.

$\min (J_1 + \mu J_2) = \|Ax - y\|^2 + \mu \|x\|^2$

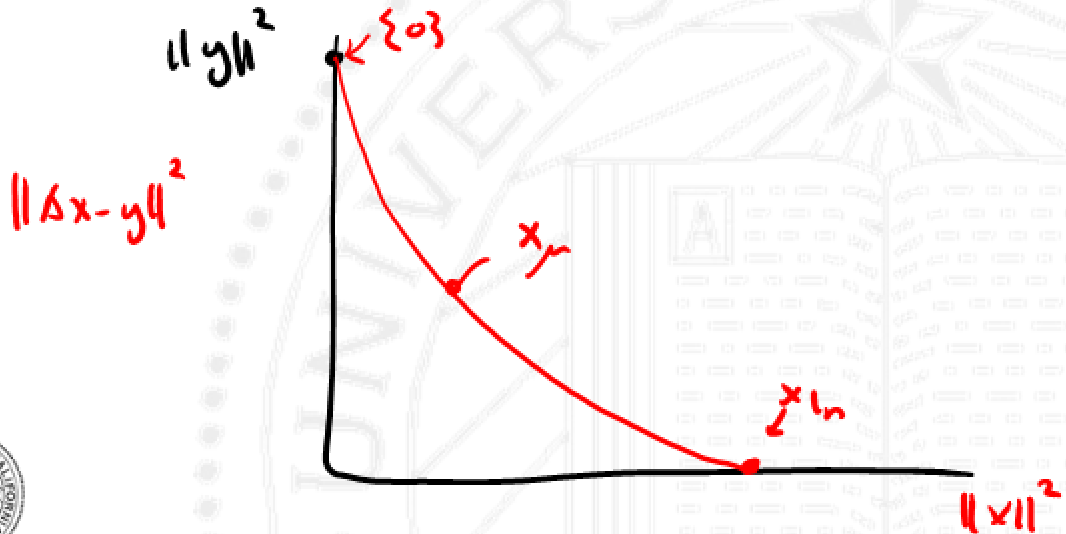
$\underline{x_\mu = (A^T A + \mu I)^{-1} A^T y}$

$x_\mu \rightarrow x_{ln}$ as $\mu = 0$.

Relation to Regularized L-S (2.2)

as $\mu \to 0$

$(A^TA + \mu I)^{-1} A^T \longrightarrow A^T (AA^T)^{-1}$ for A fat full rank



General Norm Minimization with Equality Constraints (1.3)

$$\min_x \|Ax - b\|$$
subj: $Cx = d$

↓

$$\min \tfrac{1}{2}\|Ax - b\|^2$$
subj. $Cx = d$

General Norm Minimization with Equality Constraints (2.3)

$$L(x,\lambda) = \frac{1}{2}\|Ax - b\|^2 + \lambda^T(Cx - d)$$

$$= \frac{1}{2}x^TA^TAx - b^TAx + \frac{1}{2}b^Tb + \lambda^TCx - \lambda^Td$$

$$\frac{dL}{dx} = 0 = A^TAx - A^Tb + C^T\lambda = 0$$ — solve for λ

$\downarrow$

solve for x

$$\frac{dL}{d\lambda} = 0 = Cx - d$$

General Norm Minimization with Equality Constraints (3.3)

$$\left[\begin{array}{c|c} A^TA & C^T \\ \hline C & 0 \end{array}\right]\left[\begin{array}{c} x \\ \hline \lambda \end{array}\right] = \left[\begin{array}{c} A^Tb \\ \hline d \end{array}\right] \qquad \underline{KKT}$$

$$\left[\begin{array}{c} x \\ \hline \lambda \end{array}\right] = \left[\begin{array}{c|c} A^TA & C^T \\ \hline C & 0 \end{array}\right]^{-1}\left[\begin{array}{c} A^Tb \\ \hline d \end{array}\right]$$

if (A^TA) is invertible

$$x = (A^TA)^{-1}(A^Tb - C^T\lambda) \qquad Cx = d$$

$$C(A^TA)^{-1}(A^Tb - C^T\lambda) = d$$

$$x = \left[C(A^TA)^{-1}C^T\right]^{-1}\left(C(A^TA)^{-1}A^Tb - d\right)$$

solve for $x \longrightarrow$ ugly.

$$\min_x \|Ax - b\| \quad \text{subj. } Cx = d$$

LN] $A = I \quad b = 0$

$C = A \quad d = y$

$$\begin{bmatrix} I & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} \rightarrow x_{ln} = A^T (AA^T)^{-1} y.$$

LS] $C = \{\}$

$d = \{\}$

$$\rightarrow \begin{bmatrix} A^T A & \\ & \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A^T y \\ \end{bmatrix} \rightarrow x_{ls} = (A^T A)^{-1} A^T y$$

Questions?

MIDTERM RULES

24 hour take-home 5 problems.

Open book, open notes, open my lectures
(from any year)

Open everything except,

NO LIVING HELP.

~~NO PUBLISHED SOLUTIONS~~